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INTERDEPENDENT ADDITIVE CONJOINT STRUCTURES*

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ABSTRACT

This paper deals with the use of additive value models when attributes have overlapping dependencies. An interdependent additive formulation due to Fishburn (1967) is adapted to the axiomatic system defining an additive conjoint structure (Krantz et al. [1971]). This is accomplished by replacing the independence condition defined within the additive conjoint structure with a conditional independence condition. The interdependent additive formulation is shown to be the appropriate form for value functions defined over attribute sets when certain conditions hold.

Additive Conjoint Structures

Krantz et al. (1971) define an additive conjoint structure in relation to a component set designated as $X = \times_{i=1}^{n} x_i$ where $i \in N = \{1,2,\dots n\}$, $n \geq 3$, and $\succsim$ is a binary relation on the component set satisfying five specific conditions. In particular, the structure $\langle x_1, x_2, \dots, x_n, \succsim \rangle$ is defined as an n-component, additive conjoint structure if and only if it satisfies the five axioms referred to as; a weak ordering, independence, restricted solvability, the Archimedean property and essentialism. These conditions are discussed at length in Krantz et al. (1971).

The weak ordering axiom implies that the component set satisfies connectedness and the concept of transitivity. If $x,y,z \in X$, the connectedness condition is satisfied if $x \succsim y$ or $y \succsim x$, $x \succsim z$ or $z \succsim x$ and $y \succsim z$ or $z \succsim y$. Transitivity implies that if $x \succsim y$ and $y \succsim z$ it follows that $x \succsim z$. A proof that connectedness and transitivity imply a weak ordering is provided in Krantz et al. (1971, page 19).

The $\succsim$ binary relation on $\times_{i=1}^{n} x_i$ satisfies independence if and only if for every index subset $I \subset N = \{1,2,\dots,n\}$, the ordering $\succsim_I$ induced by $\succsim$ on X for specific alternatives $x_i \in X_i$, $i \in \bar{I}$ (i.e., the index complement of I) is unaffected by the choice of those alternatives.

The restricted solvability condition is satisfied if for any index i, if $\bar{y}_i$ and $\underline{y}_i$ respectively bound y_i from above and below then whenever

$$y_1, \dots, \bar{y}_i, \dots, y_n \succsim z_1, \dots, z_i, \dots, z_n \succsim y_1, \dots, \underline{y}_i, \dots, y_n,$$

then there exists $y_i \in x_i$ satisfying:

$$y_1, \ldots, y_i, \ldots, y_n \sim z_1, \ldots, z_i, \ldots, z_n.$$

The Archimedean property requires that all strictly bounded standard sequences in the component set be finite. Given $N = \{1, 2, \ldots, n\}$ a set $\{y_i^j \mid y_i^j \in x_i, j \in N\}$ is a standard sequence on x_i if there exists $w_k, z_k \in x_k (k \neq i)$ such that $(w_k \not\sim z_k)$ for all $j, j+1 \in N$, $(y_i^j, w_k) \sim (y_i^{j+1}, z_k)$.

Essentialism requires that there exist $y_i, z_i \in x_i$ such that $y_i \not\sim z_i$ for $i = 1, 2, \ldots, n$.

Based on their definition of the n-component, additive conjoint structure, Krantz et al. (1971) show that there exist real valued functions ϕ_i on x_i, $i \in N$ such that for all $w_i, y_i \in x_i$,

$$w_1, \ldots, w_n \succsim y_1, \ldots, y_n \quad \text{iff} \quad \sum_{i=1}^{n} \phi_i(w_i) \geq \sum_{i=1}^{n} \phi_i(y_i).$$

Attribute Sets as Additive Conjoint Structures

In problems where a decision maker (DM) is attempting to evaluate riskless alternatives described over multiple attributes, it is possible to infer the existence of an additive preference function if the attribute set satisfies the requirements for an n component additive conjoint structure. If an attribute set $A = \{A_1, A_2, \ldots, A_n\}$ satisfies the four "technical conditions" (i.e., weak ordering, restricted solvability, Archimedean and essentialism), the typical procedure to show that an additive preference model holds is to test the attribute set for mutual preferential independence (MPI). Keeney and Raiffa (1976) define preferential independence as follows:

The subset of attributes $Y \in A$ is preferentially independent of the complementary set Z, iff

$$[(y', z') \succsim (y'', z')] \qquad [(y', z) \succsim (y'', z)] \quad \forall_{z, y', y''}$$

where y', y'' and z, z' are levels of Y and Z, respectively.

The attribute set A is mutually preferentially independent if every subset Y of A is preferentially independent of its complement.

In some problems, it is not reasonable to assume mutual preferential independence holds on an attribute set. One way to deal with such a situation is to redefine the attribute set to eliminate interdependencies. In some cases, this can be accomplished by the formation of attribute groupings. Such groupings would be designed to isolate interdependencies existing between individual attributes within the subgroups. In this way, it may be possible to define the subgroups such that they satisfy MPI. Attribute subgroups could then be redefined as components within an n-component additive conjoint structure if they satisfied the four technical conditions. Keeney and Raiffa (1976) discuss the use of non-overlapping multidimensional attributes in additive preference functions.

Interdependent-Additive Conjoint Structures

In some problems, it may not be possible to redefine attributes to satisfy MPI without formation of prohibitively large attribute subgroups. For example, suppose we had the eight attribute set $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and interattribute dependencies exist on the attribute pairs; (x_1, x_3), (x_1, x_5), (x_2, x_3), (x_4, x_5), (x_6, x_7). If we were to form the multidimensional three attribute set given by $\{(x_1,x_2,x_3,x_4,x_5), (x_6,x_7), (x_8)\}$, it could satisfy MPI if no interdependencies other than those stated existed.

However, it may not be practical to rank or interpret alternatives with respect to measuring an attribute level defined over five separate dimensions. One way to overcome this problem would be to form the multidimensional four attribute set given by; $\{(x_1,x_2,x_3), (x_1,x_4,x_5), (x_6,x_7), (x_8)\}$. We will designate these four attribute groups as (y_1, y_2,y_3,y_4).

The above choice of an attribute set essentially forces conditional preferential independence between y_1 and y_2 by including x_1 within each subgroup. However, for any given alternative the overlapping attribute set still satisfies the four technical conditions defining an additive conjoint structure. Given that independence is satisfied, it would be possible to show that a preference function defined on $\{y_1,y_2, y_3,y_4\}$ was additive using an approach analagous to the proof provided by Krantz et al. (1971,Theorem 13). Specifically;

$$y = (y_1,y_2,y_3,y_4) \succsim y' = (y_1',y_2',y_3',y_4')$$

if and only if;

$$\sum_{i=1}^{4} \phi_i(y_i) \geq \sum_{i=1}^{4} \phi_i(y_i') .$$

An important question concerns how to select an appropriate form for the ϕ_i in the presence of overlapping multidimensional attributes. A formulation suggested within the context of unidimensional expected

utility theory can be applied if three additional assumptions hold. These are;

1.) interval scaled measurable value functions can be assessed on multidimensional subgroups of the x_j (e.g., $f_i(y_i)$) as well as the unidimensional attributes (i.e., $f_j(x_j)$),
2.) preference interactions between the x_i are measurable by the differences of the f_i and f_j value functions.
3.) for any two alternatives y',y'' if $\phi_i(y_i') = \phi_i(y_i'')$ for $i=1,2,\ldots,m$ then $y' \sim y''$.

We will state the appropriate form of the ϕ_i due to Fishburn (1967) in the context of Theorem 1.

Theorem 1. Let $\{x_1,x_2,\ldots,x_n\}$, $n > 3$, be a set of unidimensional attributes satisfying the four technical conditions of an n-component additive conjoint structure. Suppose a set of multidimensional (possibly overlapping) attributes $\{y_1,y_2,\ldots,y_m\}$, $n > m \geq 3$ can be defined from $\{x_1,x_2,\ldots, x_n\}$ to satisfy conditional preferential independence and the four technical conditions. If assumptions 1 through 3 hold then for $y_i'',y_i' \in y$

$$y_1'',y_2'',\ldots,y_m'' \succsim y_1',y_2',\ldots,y_m' \text{ iff}$$

$$\phi(y'')=\sum_{i=1}^{m} \phi_i(y_i'') \geq \sum_{i=1}^{m} \phi_i(y_i')=\phi(y')$$

where
$$\phi_1(y_1) = f_1(y_1) \text{ and} \qquad (1)$$

$$\phi_i(y_i) = f_i(y_i) + \sum_{k=1}^{i-1} (-1)^k \cdot \sum_{1\leq j<\ldots<j_k<i} f^*\left[\bigcap_{\ell=1}^{k} y_{j_\ell} \bigcap y_i\right]$$

$$i=2,3,\ldots,m$$

where f^* is the appropriate subgroup value function for the unidimensional attribute(s) contained in $\left[\bigcap_{\ell=1}^{k} y_{i_\ell} \bigcap y_i\right]$.

The proof of Theorem 1 has two parts. The first part is to show that $\{y_1, y_2, \ldots, y_m\}$ is an m-component additive conjoint structure. With the assumption that the y_i are conditionally preferentially independent, this result could be shown using the approach of Krantz et al. (1971, pages 307-308).

The second part of the proof of Theorem 1 is to show that the ϕ_i functions are of the appropriate form. This could be done using an approach similar to that of Fishburn (1967). To see this, suppose we take the summation of $f_j(y_j)$ over 1 to m to obtain;

$$\sum_{j=1}^{m} \phi_j(y_j) = \sum_{j=1}^{m} (-1)^{j+1} \sum_{1 \le i < \ldots < i_j \le m} f\left(\left[\bigcap_{\ell=1}^{j} y_{i_\ell}\right]\right).$$

Now suppose that the above expression is partitioned into its positive and negative elements by defining the functions $g^+(i)$ and $g^-(i)$ where

$$g^+(i) = \begin{cases} 0 \text{ if } (-1)^i > 0 \\ 1 \text{ otherwise} \end{cases} \quad \text{and} \quad g^-(i) = \begin{cases} 0 \text{ if } (-1)^i < 0 \\ 1 \text{ otherwise} \end{cases}.$$

Using these functions, the assertion is that;

$$\phi(y) = \sum_{j=1}^{m} g^+(j) \cdot \sum_{1 \le i < \ldots < i_j \le m} f\left(\left[\bigcap_{\ell=1}^{j} y_{i_\ell}\right]\right) - \sum_{j=1}^{m} g^-(j) \cdot \sum_{1 \le i < \ldots < i_j \le m} f\left(\left[\bigcap_{\ell=1}^{j} y_{i_\ell}\right]\right).$$

An alternative means for stating this would be;

$$\phi(y) + \sum_{j=1}^{m} g^-(j) \cdot \sum_{1 \le i < \ldots < i_j \le m} f\left(\left[\bigcap_{\ell=1}^{j} y_{i_\ell}\right]\right) = \sum_{j=1}^{m} g^+(j) \cdot \sum_{1 \le i < \ldots < i_j \le m} f\left(\left[\bigcap_{\ell=1}^{j} y_{i_\ell}\right]\right). \tag{2}$$

To prove Theorem 1, we can show that the above two expressions are equivalent with respect to y_i for $i=1,2,\ldots,m$. To do this, we define the left hand side of (2) with respect to y_j. Dropping the value functions for the moment and concentrating on the actual preference states, the left hand side of the above expression with respect to y_j is given by;

$$y_j + \sum_{k=1}^{m} g^-(k) \cdot \left\{ \sum_{\substack{1 \le i < \ldots < i_k \le m \\ j \in \{i, \ldots, i_k\}}} \left[\bigcap_{\ell=1}^{k} y_{i_\ell}\right]_j + \sum_{\substack{1 \le i < \ldots < i_k \le m \\ j \notin \{i, \ldots, i_k\}}} \left[\bigcap_{\ell=1}^{k} y_{i_\ell}\right]_j \right\}$$

where $[y_i]_j$ represents the overlap of y_i with y_j. The preference state on the right hand side can be described as;

$$\sum_{k=1}^{m} g^+(k) \cdot \left\{ \sum_{\substack{1 \le i < \ldots < i_k \le m \\ j \in \{i, \ldots, i_k\}}} \left[\bigcap_{\ell=1}^{k} y_{i_\ell}\right]_j + \sum_{\substack{1 \le i < \ldots < i_k \le m \\ j \notin \{i, \ldots, i_k\}}} \left[\bigcap_{\ell=1}^{k} y_{i_\ell} \bigcap y_j\right]_j \right\}$$

Taking the component for k=1 out of this expression from the summation where $j\epsilon\{i,\ldots,i_k\}$ yields;

$$y_j + \sum_{\substack{k=1\\k>1}}^{m} g^+(k)\cdot\left\{\sum_{\substack{1\le i<\ldots<i_k\le m\\ j\epsilon\{i,\ldots,i_k\}}}\left[\bigcap_{\ell=1}^{k} y_{i_\ell}\right]_j + \sum_{\substack{1\le i<\ldots<i_k\le m\\ j\not\epsilon\{i,\ldots,i_k\}}}\left[\bigcap_{\ell=1}^{k} y_{i_\ell}\cap y_j\right]_j\right\}.$$

With this, the first term in the expressions for the left and right hand sides of expression (2) cancel, and it remains to show that;

$$\sum_{k=1}^{m} g^-(k) \sum_{\substack{1\le i<\ldots<i_k\le m\\ j\not\epsilon\{i,\ldots,i_k\}}}\left[\bigcap_{\ell=1}^{k} y_{i_\ell}\right]_j = \sum_{\substack{k=1\\k>1}}^{m} g^+(k)\cdot\sum_{\substack{1\le i<\ldots<i_k\le m\\ j\epsilon\{i,\ldots,i_k\}}}\left[\bigcap_{\ell=1}^{k} y_{i_\ell}\right]_j \quad (3)$$

and,

$$\sum_{k=1}^{m} g^-(k) \sum_{\substack{1\le i<\ldots<i_k\le m\\ j\epsilon\{i,\ldots,i_k\}}}\left[\bigcap_{\ell=1}^{k} y_{i_\ell}\right]_j = \sum_{k=1}^{m} g^+(k) \sum_{\substack{1\le i<\ldots<i_k\le m\\ j\not\epsilon\{i,\ldots,i_k\}}}\left[\bigcap_{\ell=1}^{k} y_{i_\ell}\cap y_j\right]_j \quad (4)$$

In (3), notice that $\cap y_j$ has been deleted from $\bigcap_{\ell=1}^{k} y_{i_\ell}$, as a result we can also delete $i_\ell = j$ and correspondingly reduce k by one to get the right hand side of (3) equal to;

$$\sum_{k=1}^{m} g^-(k)\cdot\sum_{\substack{1\le i<\ldots<i_k\le m\\ j\epsilon\{i,\ldots,i_k\}}}\left[\bigcap_{\ell=1}^{k} y_{i_\ell}\right]_j . \quad (5)$$

If m is odd, then k=m in the right hand side of (3) becomes k=m-1 in (5), otherwise the maximum value of k on the right hand side of (3) is m-1 which goes to m-1 in (5). However, if m is even, then $j\epsilon\{i,\ldots,i_m\} = \{1,2,\ldots,m\}$ so that (5) has no terms when k=m thus establishing (3).

In (4), the right hand expression has $\cap y_j$ added onto $\bigcap_{\ell=1}^{k} y_{i_\ell}$ so that $i_\ell=j$ can be added to the other i_ℓ and the index k can be incremented by one to obtain;

$$\sum_{k=1}^{m} g^-(k)\cdot\sum_{\substack{1\le i<\ldots<i_k\le m\\ j\epsilon\{i,\ldots,i_k\}}}\left[\bigcap_{\ell=1}^{k} y_{i_\ell}\right]_j . \quad (6)$$

In (6), we need not consider k=m+1 if m is odd since $g^+(k)=0$ in (4) and this term drops out. Otherwise, k=m-1 in the right hand side of (4) goes into k=m in (6) establishing (4) and completing the proof.

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