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FINAL REPORT

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Preface and Summary

This study recognizes law enforcement as an industry replete with problems suitable for operations research and economic analysis. The application of this body of theory is made more difficult because the "commodities" that are a part of the cost and payoff in this industry are not market traded. Work in this area involves the search for proxies and the construction of shadow prices.

We chose to investigate the relationship between the time spent on the preventive patrol of an area, the number of crimes committed, and the ratio of arrests to reported crimes. Our model is based on rational criminal behavior which responds to the expected costs imposed on the criminal. Preventive patrol is designed to raise the expected costs to the criminal by raising the probability of capture. This empirical analysis is based on the assumption that no change in preventive patrol strategy takes place during the data collection period. Changes in inputs can then be properly measured by changes in time devoted to preventive patrol. We chose retail store burglaries as a crime likely to be the result of rational planning. The statistical procedures were developed and tested with sample data. Lack of actual data prevented a field test.

A statistical scheme for the analysis of the burglary detective operation was also included. It concentrates on identifying the attitudes of a burglary that are most important for its solution. Again, lack of data prevented a test.

Turning from an attempt to describe and evaluate some of the current activities of police patrol and detective forces to an attempt to develop better patrol strategies brings us to the second major portion of the report. To be effective, preventive patrol must deter crime by its very presence or by its ability to make arrests. We ignore the first factor except as a constraint on minimum acceptable length of patrol. Making arrests increases the arrest to crime ratio which serves as an estimate of the probability of capture to the potential criminal thereby leading to a deterrent effect.

In order to effect an arrest or even to detect a crime in progress, a preventive patrol device must come in visual contact with the crime in progress. Concentrating on a patrol force operating in squad cars, this means that a car must pass the scene of the crime while it is in progress and the officers in the car must detect the crime. In the parlance of search theory, this is called achieving space-time coincidence. The probability of achieving space-time coincidence depends on the distribution of crime in the area under surveillance, the patrol route taken and the number of cars involved. The models developed in the second part of the report concentrate on the use of search theory to design an optimal preventive patrol strategy. They are constructed with many of the institutional limitations on police activities included and are of potential applicability to the police. At the same time they represent a contribution to the search theory literature.

The difficulty in communicating operations research and economic research on real problems to the people who might most directly benefit from the research is certainly not peculiar to studies of police problems. Usually, the trouble is blamed upon the researcher's penchant for what is labeled technical jargon. But the fault lies on both sides. One cannot hope to benefit from new techniques if he is unwilling or unable to make the difficult and time-consuming effort necessary for at least a rudimentary understanding of them. It is not obvious that the new techniques will, in fact, be of benefit. Police administrators would be foolish indeed to accept them (or to reject them) without a trial. Fruitful trials will not be made unless the analyst understands the rudiments of the administrator's problem and the administrator understands the rudiments of the analyst's techniques.

Techniques developed for one use usually cannot simply be applied to other uses. Abstract models of police problems must be formulated and techniques developed for their solution. This process should not be expected to yield directly applicable results immediately. Indeed, the development of a sound body of the management sciences in police activities may well be slowed by insistence of funding agencies for immediate applicability. If science and technology are to be of real aid in solving the problems faced by police administrators, a necessary condition would seem to be the existence of a body of technically well-trained people who are interested in "police problems." The main contribution of this project may well be in this area.

This report was written by David Olson and Gordon Wright, responsible for Part II and by John Mayne and Arthur Hurter for Part I. Alan Karr and Alan Cohen helped in the preliminary work.

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PART I: POSITIVE ANALYSIS

Introduction

This project attempts to evaluate the applicability of some management science analysis, particularly economics and operations research, to police resource allocation problems. While recognizing the interdependencies between police activities and many other aspects of the urban environment, the study assumes that a meaningful allocation problem exists and can be formulated for portions of the police activities.

The allocation of law enforcement resources implies the precedent problem of defining the appropriate size of the law enforcement industry. The product of this industry is the prevention of criminal behavior. The social gain from crimes prevented is the value of harm forestalled. If this could be done at zero cost, social welfare would be maximized by preventing all crime. Since crime prevention consumes resources that have alternative uses, the scale of law enforcement should be pushed to the margin where harm forestalled is equal in value to resources employed to achieve it. Attempts to improve police (or society's) ability to prevent crime must assume that criminals are responsive to costs imposed upon them. The most important components of cost are the anticipated value of punishment and the probability of some punishment.

Law enforcement is an industry replete with interesting allocational problems. Economics and operations research may be very useful in solving these problems but their application is more difficult because the "commodities" that are part of the cost and payoff in this industry are not market traded. Work in this area, then, involves the search for proxies and the construction of shadow prices.

For each offense, there may be a loss to the victim and a gain to the offender. In addition, there is a loss to society because the crime took place. The net loss of these items is the social loss from the crime. In addition, there is a cost of police activity which is determined by the manpower, materials and capital utilized. These resources primarily influence the probability of conviction and the cost of police activity per offense. A major share of police activities are used in the patrol function. This is not normally directed toward a single offense or even type of crime. Here, on the cost side, the production relationships are complicated by joint products.

We assume that a person commits an offense if the expected utility to him exceeds the utility he could get by using his time and other resources in other activities. Some persons become criminals, therefore, not because their basic motivation differs from that of other persons but because their benefits and costs differ.

Preventive activities should be directed toward lowering the benefits of criminal activity while raising the opportunities for benefits from other activities. Criminal justice activities are aimed at lowering the benefits of crime, primarily by increasing the probability of conviction and by assigning punishment to convicted offenders. There is a tendency to attribute all kinds of criminal behavior to ill-defined social forces and to argue that the criminal could not avoid commission of crime. Since the criminal does not then respond to the "costs" imposed on him, i.e. he does not make a rational choice, the basis for preventive police measures is largely removed. While recognizing that many criminal acts such as murder and rape are not the results of rational thought processes, (the so-called crimes of passion), this report adopts the point of view that some crimes (e.g., burglary) are the result of rational thought processes. At the very least, it assumes that the observed behavior of some types of criminals can be predicted on the assumption that criminals behave as if they were sensitive to a system of rewards and costs. If all crimes were "crimes of passion" the police role would be one of apprehension of offenders. This has a deterrent effect to the extent that criminals are removed from society but it does not have deterrent influence on potential offenders.

The measurement of the influence the activities of a police force have on the number and type of crimes committed is a difficult and largely unsolved empirical question. Yet, it is at the heart of any analysis of police resource allocation problems. Recognizing the extreme difficulties involved, this report concentrates on burglary since it is least likely to be purely a "crime of passion". In particular, we concentrate on retail store burglary. Of course, the joint product character of the police patrol activities never ceases to be a troublesome, complicating factor.

In addition to attempting to measure the influence of "preventive" patrol on burglary rates, a scheme for more effectively organizing preventive patrol activities is developed.

General Analysis

This report distinguishes three elements that determine the rewards of a particular burglary as the rational burglar would view them. They are:

(i) Rewards from alternative uses of resources

(ii) Rewards from the burglary under consideration if successful

(iii) Probability that the burglary in question will be successful

Together these three elements determine the expected net return to a particular burglar from the commission of a particular burglary. In the second quarterly report, the relationships between exogeneous variables and the three elements just discussed were treated in detail. Only a brief description of the six equation model is included here.

In a completely disaggregated form, each individual and each retail establishment would have to be identified. Some aggregation seems in order. With an eye toward empirical usefulness, we chose aggregations that made use of groupings already established. Individuals are grouped into neighborhoods or census tracts according to their residence addresses. Retail establishments are grouped into police districts or beats according to their location.

The key variable is the expected net return from a burglary in district k by an individual living in the j-th neighborhood. This dependent variable [${}_kQ_j$ in equation 2 of 4th quarterly report] depends upon: the expected gross return from a retail burglary in the k-th retail district in a particular time period [R_t^k], the probability that the burglary will be successful, and the value of alternatives foregone by individuals living in j if they undertake a burglary in k [$g(I_t^j)$].

The cost of opportunities foregone by individuals residing in neighborhood j depends upon what may be thought of as the environmental and demographic factors appropriate for the j-th neighborhood. These <u>may</u> include median age, median years of education, proportion non-white, age of residential buildings, median assessed property value per family, median family size and a measure of the rate of unemployment $[Z_i^j]$. It is assumed these exogeneous variables can be combined to yield a measure of the average tendency toward burglary (crime) of residents in neighborhood j. The relationship between the exogeneous, environmental and demographic factors and the tendency toward burglary must draw upon research in economics, political science, sociology and psychology which is not presently available. For purposes of this report, the existence of such a relationship is assumed. It is also assumed that police activities do not influence the environmental and demographic variables in a neighborhood. Consequently, for our purposes, the value of alternate opportunity to residents of neighborhood j is taken as given.

The gross return from a successful burglary is directly related to the character of the establishment, and the shopping district being considered. It is influenced by the same entities that influence the economic prosperity of the retail area. Location theory is used to identify the variables of importance and to delineate the market area for each retail district. Once the market area is defined, then variables that measure its purchasing power and the frequency with which customers visit the area can be used to estimate the anticipated gross return from a successful burglary. This relationship is discussed at length in the third quarterly report and will be discussed in the next section of this report.

The final element affecting the expected return from a burglary in district k by an individual from neighborhood j is the probability of its success. It is assumed that a burglary is a success if something of value is taken and the burglar is not captured. As far as the patrol force is concerned, the burglary is a success if something of value is taken and the burglar is not captured in the act, or as the result of immediate search and pursuit. In any case, a burglary is either a success or a failure. Thus the probability of a successful burglary is one minus the probability of capture. The probability of capture depends upon private devices such as burglar alarms, watchmen and dogs. It also depends upon police activities. Taking the private efforts as given, the probability of capture can be considered as a measure of police output.

The probability of capture can be estimated as the ratio of the number of burglary arrests made in district k over some period of time to the number of reported burglaries in district k over the same period of time. We assume that burglars from neighborhood i are equally adept as burglars from neighborhood j so the probability of capture is the same for all. The probability of capture will be treated as a variable primarily influenced by police activities. The variables depicting police activities are the major control variables in the overall model.

The number of burglaries in retail district k by people from neighborhood j in time period t+1 depends upon the expected returns from burglaries in district k by people from neighborhood j in time period t. The total burglaries in district k is the sum over all neighborhoods. Thus, by raising the probability of capture through, say, a more efficient allocation of resources, the police can reduce the number of burglaries because they reduce the expected net return from a burglary.

If our goal were merely to predict burglaries or any crimes in the next time period, then a straightforward extrapolation of time trends will probably yield accurate predictions. In fact, this has proven to be the case in both the St. Louis and Chicago police departments. However, we are interested in more than just the prediction of quantities of criminal activity. We want to relate police activities to the level of crime, at least to the level of burglary, as has just been described.

The Gross Return Function

The return function was the major topic of the third quarterly report. The gross return from a successful burglary depends upon the economic characteristics of the retail establishment selected. Among other things, the economics of the store in question depends upon the types of goods it sells. The dollar sales of stores selling goods of type g in retail district k in time period t (${}_{t}R^{g}{}_{k}$) depend upon the probability that people residing in neighborhood i shop for commodity g in district k. The dollar sales in time period t from goods of type g in district k from people living in i, depend upon the number of people living in neighborhood i and the per capita expenditure in time period t on goods of type g.

The probability that a consumer living in neighborhood i would shop for good g in retail district k (P_{ik}^{g}) is a function of retail district k and its accessibility. If S_{j}^{g} is the number of stores in district j that sell good g and T_{ij}^{g} is the adjusted travel time (people will travel longer distances for some goods than for others) from i to j, then:

$$P_{ik}^{g} = (S_{k}^{g}/T_{ik}^{g}) \,/\Sigma_{j}\, (S_{j}^{g}/T_{ij}^{g})$$

This probability multiplied by the total dollars expended on goods of type g per capita times the number of people in neighborhood i yields the expected dollar sales for all stores in district k, selling good g, from people living in neighborhood i. Summing over all neighborhoods and goods yields estimates of dollar sales for the stores in district k. This model has apparently been used, with the aid of a computer (see 3rd quarterly report), to estimate retail dollar sales.

To apply this model, the retail centers, demographic classes (if different from neighborhoods), neighborhoods and types of goods must be specified and defined. The data required and sources for these data are developed in the third quarterly report. The primary sources are the U.S. Census of Population and Housing, the U.S. Bureau of Labor Statistics Consumer Expenditures and Income and the U.S. Census of Business - Retailing.

There can be little doubt that the rewards from a successful burglary depend upon the economic characteristics of the store being burglarized in much the same way that the store's economic well-being depends on the same characteristics. From the point of view of the potential burglar, the selection of a location for his business is much like the decision of a retail merchant selecting a location for his establishment. Thus, we considered location analysis for aid in formulating our model. One of the surprising features of our work is the discovery that very few useful models of retail location appear in the open literature. The research discussed in the third quarterly report represents the extent of useful work in retail location theory known to us. Thus, one of the original reasons for attempting to use location theory seems much weaker now than when the project proposal was written. The idea of adapting a reasonably complete, useful theory of retail store location to the choice of sites for burglary fails because of the character of the published work on retail location decisions.

The idea is, in the author's opinion, still worth pursuing, although it is clear that some purely theoretical work is required before useful models can be developed. In terms of a particular retail store, a market area, not for customers, but for potential burglars might be developed. The sensitivity of the area to changes in store characteristics (e.g. types of goods) would be of interest and could help to explain the burglar's choice of burglary sites. A complete model of choice for burglars of different demographic and environmental background would include aspects of the alternatives available to an individual. It would thus include two of the three basic elements of choice for a burglar, rewards from alternative uses of resources and rewards from the burglary under consideration, if successful. The spatial structure of an urban area or a part of an urban area could then have a direct, explicit influence on the choice of burglary sites. If successful, such a study would be of interest to planners and designers.

Police Activities

The bulk of our study is concerned with the activities of the police and their influence on the number of crimes committed (reported) and the probability that the criminal is captured. If the number of burglaries is taken as given for time period t, then police effectiveness against burglary is measured by the proportion of these burglaries foiled through capture of the burglar. This effectiveness, in turn, should influence the number of burglaries in future periods by altering the expected net return to the potential burglar.

Thus, we are concerned with the relationships between the inputs utilized by the police, and the manner in which they are used on the one hand and the proportion of crimes that result in capture and in the number of crimes committed on the other. In short, the production function relationships for police activities are being sought.

Our search for a production function is complicated by the manner in which most police departments operate. If we abstract from the detective, traffic and public relations activities, we are left with the patrol activity. With some notable exceptions, the patrol force performs two duties: preventive patrol and response patrol. The preventive patrol attempts to reduce the burglar's estimation of his chances of success based on police presence. The response patrol attempts to deter crime through the capture of criminals after a call for service has been placed.

In addition to performing the joint function of response and preventive patrol, the police patrol force is on duty against all crimes. The design of a patrol force must take into account its possible effectiveness against murder, robbery, rape, burglary, auto theft, etc. The optimal tactic to employ against burglary might be a very poor one to employ against murder.

Perhaps the least sophisticated format suggested for the evaluation of an industrial production function is input-output analysis. Here, the firm is treated as a black box which takes in inputs and transforms them into outputs. The actual transformation between inputs and outputs is ignored. The inputs and outputs are identified and assumed to be related through one or more linear relationships depending on the number of outputs identified. The coefficients of the linear relationships are empirically determined using cross-section or time series data. Simple relationships of this sort are purely descriptive. They provide a convenient and sometimes useful means of summarizing the history of an organization (time series) or the current "state of the art" (cross-section). Unless the "firms" being studied are using optimum tactics, the results of the study do not represent the production functions of economic theory which assume that technological efficiency has been attained.

When we attempt to determine an input-output type of production relationship for police patrol activities, our difficulties begin with a definition of the appropriate measures of inputs and outputs. In view of our earlier discussion, the measure of output selected is the ratio of burglary captures to burglaries in a particular area for a particular period of time. Another alternative is to measure the total number of arrests for all crimes and divide it by the total number of crimes. This would yield an estimate of the overall police output. (We recognize the difference between the number of actual crimes and the number of reported crimes. In using the latter, we are tacitly assuming that the ratio of the two remains constant as police activities are altered.)

We have decided to concentrate our attention on retail store burglaries because we feel that this is one of the crimes upon which the activities of a police preventive effort may have some effect. For our purposes, then, the relevant output measures are the number of (reported) retail burglaries in district k in time period t and the number of "unsuccessful" retail store burglaries in the same district at the same time. When only the patrol force is under examination, we consider a burglary as unsuccessful when the burglar is caught within a short period, say one hour, after the crime is observed or reported. If the criminal is captured later by the detective force, the crime is still a success as far as the patrol force is concerned.

Our choice of output variables referring only to retail burglary conceptually dictates that the input variables also be directly related to patrol activities against retail store burglary. However, the patrol activities of the police are not usually directed against specific crimes. A police patrol unit, when not answering a call for service, patrols its beat on the look out for all kinds of crime. The presence of the police alone may serve as a deterrent to a burglar since it may decrease his estimate of the probability of success. However, such preventive patrol measures may have a similar deterrent effect on crimes other than retail burglary. Even so, the time spent on preventive patrol can be considered a direct input in the retail burglary function. If this preventive patrol effort does deter other types of crime, this in no way lessens its effect against retail burglary. In a sense, preventive patrol functions something like a public good.

Police patrol resources are usually assigned to beats and these beats differ in geographic identity from the retail districts described by the Bureau of the Census. Assuming no major change in patrol technology, the measure of input must be related to the number of beats that include the particular retail district and the frequency with which the beat cars patrol the retail district portion of each beat. It may be advantageous to distinguish between two-man cars, one-man cars, motorcycle patrol, foot patrol, and T.V. surveillance. Any police department run with a modicum of efficiency should regularly keep and record data measuring the use of each of these types of inputs on preventive patrol.

In addition to the preventive patrol activities, another means of increasing police output is for police resources to respond quickly to calls for service. Therefore, the response patrol input against retail burglary is the average proximity of response forces to the retail district of interest. A common measure of such an input is the average response time. Police departments should certainly have data on how long it takes them to answer calls for service.

The private resources devoted to prevention are more difficult to measure. Two approaches are suggested in the fourth quarterly report and they will not be repeated here. Data on private prevention efforts will be difficult to obtain and will probably require sample surveys. For purposes of this study, assume that private efforts will not be directly influenced by changes in police activity and can be considered a constant. Of course, this assumption cannot be valid over long periods of time and is employed primarily as a convenience.

If we think of holding all elements of the burglary generation relationship constant except those related to the probability of a successful crime, then police activities should influence the number of retail burglaries reported in a district and over a time period in a measurable way. Over relatively short time periods, this is probably not a bad assumption. The return from a successful burglary, depending as it does on the characteristics of the store and the neighborhood in which it is located will probably not change significantly over, say, a six month period (unless, of course, the management changed). Surely, the characteristics of the neighborhood in which the burglar lives and his alternative opportunities will not change a great deal over the same period. If these elements of the relationship yielding the expected return from a burglary are constant then the expected return becomes a function of the probability of a successful burglary alone. With private preventive activities taken as fixed, the probability becomes a function of police activities alone. Since the number of burglaries (reported) in time period t+1 in district k depends upon the expected return from such a burglary in period t, we can state an empirically testable hypothesis: Increases in police patrol activities in period t reduce the number of retail burglaries committed in period t+1.

In this context, increases or reductions mean changes in quantity of the resources used in the same manner as presently employed. Hopefully, this hypothesis can be tested using readily obtainable data from police department records. Until such records and cities are identified, the details involved in identifying retail districts, neighborhoods, beats, etc. cannot be specified. The result of such an analysis would be to discover the sensitivity of the number of retail burglaries, and of the unsuccessful burglary to reported burglary ratio, to the inputs of patrol activities measured in terms of response time and time spent on preventive patrol.

Given a knowledge of the technology and procedures employed in any city, the costs of the police inputs could be determined. Notice that the relationship between the unsuccessful to total burglary ratio and the two "time" inputs is a production function relationship. It is a convenient means of summarizing the observed modes of operation. However, it is not detailed enough for police resource allocation decisions. It may be that two, two-man squad cars operating on a beat provide a particular level of response time and preventive patrol time. The influence of these levels of response time and preventive patrol time may yield a particular result with regard to the output variables. However, there may be more efficient ways of obtaining the <u>same</u> values of response and preventive patrol time that cost less than the two, two-man cars per beat procedure. Nevertheless, the effect on the output variables would be the same.

At this point, the direction of future research should be clear. Economic and operations research analysis should be employed in an attempt to find the efficient means of providing any predetermined combinations of response time and preventive patrol time. Econometric and statistical analysis should be employed to develop the sensitivity of the number of burglaries committed in district k during period t and the ratio of unsuccessful to total burglaries in district k during t. Our work has concentrated part of its resources on the economic and operations research quest for efficiency and another portion on the econometric-statistic attempt at establishing an empirical production function for testing sensitivity.

As will become clear in succeeding sections, the optimal design of a patrol force is complicated by its dual role as responsive and preventive. Here, it is convenient to specify either a minimum acceptable preventive patrol time or a minimum acceptable response time and to maximize the other subject to a budget constraint. Parametric techniques then permit the establishment of tradeoffs between response time and preventive patrol time. This is discussed further below. In some contexts of importance, the response time and the preventive patrol time are complementary inputs. Increasing one tends to increase the other. Here, the design of an efficient response force would lead to an effective preventive force as well. However, this advantageous situation will not always obtain and, in general, tradeoffs between patrol forces designed on the basis of efficient response and those based on efficient preventive patrol must be made.

The Burglary Detective Activity

In reviewing some of the efforts of the burglary detective division we were struck with the lack of analysis of retail burglary cases turned over to the detective division by the patrol division. An empirical analysis of burglary and detective activities is proposed in the fifth quarterly report and is summarized below.

The objective of this research is to identify those attributes of a retail (or commercial) burglary that lead to the clearance of the case by detectives. The results would also help to determine what kinds of information about a crime are worth obtaining and what kinds do not aid in efforts to clear a case. The retail burglary cases turned over to the detective division must be considered successful burglaries from the point of view of the patrol force.

This analysis assumes that no change in the procedures or abilities of the detective force took place over the time period under analysis. The basic source of data is the report filled out by the patrol officer who initially answers the call for service. This report is supplemented by the detective who takes over the case. In Chicago, for example, this report is called the burglary case report. As an initial analysis, we include as attributes of importance only those items listed on the Chicago Burglary Case Report. These include: an identification of the location of the store and the beat on which the burglary occurred, the kind of store (e.g., liquor, cleaning), was a safe involved, dollars of cash taken, dollars of jewelry taken, dollars of furs taken, dollars of clothing taken, dollars of other goods taken, number of offenders, time of burglary, race of victim, man-hours of detective force effort until case cleared or "given up". These attributes would be measured and recorded for each case turned over to the burglary detective unit, in district k during period t. They can be thought of as independent variables. The dependent variable, a binary variable, simply identifies each case as cleared or uncleared. On cleared cases, additional information can be obtained from the burglar: residence of offender, sex of offender, age and race of offender.

We want to be able to classify burglaries, on the basis of their attributes as either cleared or uncleared. Further, we want to get some measure of the relative importance of the different attributes in determining whether or not a case is cleared. We want to be able to take the values of the attributes of a burglary case turned over to the Detective Division and to predict whether or not this case is going to be cleared. The standard statistical technique to use for this kind of classification is discriminant analysis. However, the application of discriminant analysis to this problem is not without difficulty. The theory of discriminant analysis was developed for continuous independent variables (attribute measures). However, many of the attributes are best measured in discrete terms. For example, a store is either a liquor store or it is not (i.e., this variable is 0 or 1). Furthermore, we are measuring these variables on arbitrary ordinal scales (e.g., rather than 0 or 1 we could have used 1 or 10) rather than on a cardinal scale. In this situation, parametric statistical tests, which use means and standard deviations (i.e., which require the use of arithmetic on the original scores) theoretically ought not to be used with data in an ordinal scale such as attribute scores. Discriminant analysis does involve means and standard deviations.

Some effort was spent during the fifth quarter attempting to thoroughly learn discriminant analysis and to find means of adapting it for the burglary detective problem. A report was prepared and summarized in our fifth quarterly report. Finally, it was decided to employ a linear discriminant function computed as if the independent variables employed were continuous.

Our trial area was to be the 20th police district in the City of Chicago. Other studies of burglary had already been performed in this particular district. In addition, some of the data obtained for our burglary detective division study could also be employed in our study of patrol forces and burglary to be discussed further below. The measures of the attributes listed earlier can be obtained from the Burglary Case Reports. In Chicago, this information is routinely punched into the "burglary case report summary tape" and should be readily available. Data were to be collected on cases that were about one year old in order to avoid some of the ambiguity caused by a case being neither cleared nor "given up" as unsolved because it was still being worked on. The summer period was selected as more interesting and we tentatively selected the period May 1969 through October 1969.

The 20th district in Chicago is primarily a combination of residential and commercial enterprise. From the middle of 1968 to the middle of 1969 there were 592 reported commercial burglaries. There were approximately 17 commercial burglaries during this year per linear street mile. A computer program for linear discriminant analysis is available at the Vogelback Computing Center of Northwestern University. Our development of a statistical analysis technique has progressed to the point where it should now be tried with "real" data. The Chicago Police Department has decided against making these burglary case reports available to us. Yet we feel that the analysis suggested could yield interesting results in terms of the allocation of resources between the patrol division and the detective division. It might also point to a new form for a burglary case report emphasizing the data that seem more important in capturing a burglar.

Patrol Resources and Burglary

In this section we return to our earlier discussion relating the number of burglaries in a district during a period of time, the number of unsuccessful burglaries in that district over the same period of time and the activities of the police patrol forces. Again, the objective is descriptive rather than prescriptive. When we talk about changes in police inputs, by and large we are talking about more or less of the same technology employed during the period over which data were collected.

From the point of view of the police administrator, the police inputs are men (man-hours) and capital equipment (squad cars). In terms of our earlier discussion, these are converted to inputs like response time and time on preventive patrol. Unfortunately the technology that relates man-hours and squad cars to time on preventive patrol and response time is not well known. It is in making this transformation as efficient as possible that economic analysis and operations research find their major role. The focus of our attention is again on commercial burglaries. The general feeling is that more effective police patrol will reduce the number of burglaries committed.

The patrol force of the Chicago Police Department is not divided into a response force and a preventive force. To some extent, the St. Louis force is divided in this manner. In Chicago, each patrol car performs both preventive and responsive tasks. Of course, the response function takes top priority so that the time devoted to preventive patrol is a residual. Further, the data will reveal that there is more preventive patrol done during those parts of the day with the least crime. Clearly, this result cannot be credited to preventive patrol activity. There seem to be certain hours of the day that are preferred by burglars. Recall that the patrol cars answer calls for service of all kinds so that high crime portions of the day are not necessarily high commercial burglary times.

We decided to design an econometric study of some police beats in the 20th Police District in Chicago where burglary is a crime of major magnitude. A particular block of hours in the day (e.g., 8 P.M. to 2 A.M.) was selected as the most active hours for burglars. The plan was to collect data on the number of commercial burglaries, the number of commercial burglary arrests and the time devoted to preventive patrol on several beats. The data would be recorded, for the chosen hours and beats, on a daily basis over some rather long period of time, say six months. The initial time periods might be identified by examination of the records of private burglar alarm companies. One of the difficulties with ordinary police records on burglary is that often the time of occurrence is unknown. The preventive patrol efforts of the beat car are, of course, not all concentrated in commercial areas unless the entire beat is in a commercial area. Thus, only a fraction of the time spent on preventive patrol can be considered devoted to prevention of retail burglaries. This difficulty can be alleviated by assuming that the ratio of time spent patrolling the commercial areas to total time spent on preventive patrol is equal to the ratio of miles of commercial streets to the total street miles on a beat. However, since we are assuming that police patrol tactics do not change over the data period, it is only necessary to measure the time spent on preventive patrol.

Consider a particular beat and a period of time during each day. Then the data required on a "daily basis" can be summarized as:

(i) The number of commercial burglaries [obtained by sorting the usually collected burglary data]

(ii) The number of on-view arrests (defined to include arrests made immediately after the crime) as a measure of unsuccessful burglaries [Police burglary data]

(iii) The number of men, patrol units and the time they spent on preventive patrol. [Police records of activities of patrol units]

In the Chicago Police Department, for example, the Radio Dispatch Summary Tape could be sorted by beat to give almost a minute by minute account of the reported activities of each patrol unit. In addition, they would provide the case report number and dates of any commercial burglaries on the beat in question. The case numbers can be used to identify the Burglary Case Reports of interest. These yield an indication of the type of burglary, the time of the crime and whether or not the burglary was a success.[1]

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1. An effort was made to obtain the Burglary Case Report Tapes and the Radio Dispatch Summary Tapes from the Chicago Police Dept. However, they were not available to us. There is the possibility that similar data could be obtained from the St. Louis Police Dept. Our initial contacts indicate a willingness on their part to supply data but of course a specific proposal would have to be approved by their Board of Police Commissioners. However, they sometimes split their patrol functions into preventive and response forces. Further, they apparently don't have anything quite like the Chicago Radio Dispatch Summary Tapes making acquisition of data on the hour to hour activities of the patrol units uncertain. We were able to contact the American District Telegraph Company and their records indicate that almost all commercial burglaries, covered by their alarms, took place between 10 P.M. and 3 A.M. on weeknights.

Now suppose the data desired were obtained. We would have three time-series of variable values on each pre-selected beat.

(i) N_t, the number of commercial burglaries each day. They are assumed to have occurred during the "critical hours" e.g., 10 P.M. to 3 A.M., see footnote 1.

(ii) C_t, the number of "on-view" arrests (i.e. the number of unsuccessful burglaries from the point of view of the patrol force) that occurred each day.

(iii) T_t, the time spent on preventive patrol each day.

One of the difficulties encountered with commercial burglaries is that they often occur unnoticed by the victim until the next business day. Thus, it is difficult to specify their time of occurrence. This has led us to the definition of N_t above. However, a useful treatment of burglaries on weekends is yet to be developed.

Further difficulties stem from the magnitude of the numbers involved. Many elements influence the rate of crime in a city. By collecting data on a day to day basis and by beat on a single type of crime we tend to hold many of these things constant thus permitting concentration on the effect of police patrol activities. While no one will claim that crime rates in Chicago are too low, the values obtained for the N_t variables seldom exceed 2 and most often are zero. Of course, the number of unsuccessful commercial burglaries as measured by "on-view" arrests is even smaller. Thus, we expect that observed values for N_t, if measured for a beat, by day, for commercial burglary would be 0, 1, 2, and 3 with very few 3's. This characteristic of the data is a key factor in our choice of statistical techniques.

In most empirical work, the fine details of the analytical procedures cannot be specified until some preliminary investigations have been made with the data in hand. At this time we feel that the following methods should be applicable to the data just discussed.

Single classification analysis of variance

Here, the intent is to classify the observed commercial burglaries according to the intensity of the preventive patrol on the night of occurrence. This kind of classification presupposes the prior designation of levels of patrol activity into categories e.g., light, medium and heavy. Whenever categories of this type are suggested, the problem of defending them against those who think they should be more or less numerous or perhaps based on different criteria must be faced. Breaking an essentially continuous variable e.g., time spent on preventive patrol, into discrete categories always suggests the possibility that other category designations might be more useful. Questions of this nature can only be resolved after the data are at hand.

For purposes of exposition, assume k categories have been defined. Next, define the random variable x_i as the number of commercial burglaries that occurred when preventive patrol of category i (i=1,...,k) occurred. If we adopt the categories light (i=1), medium (i=2) and heavy (i=3) then x_2 is the number of commercial burglaries that occurred on a beat during a time period with medium preventive patrol. Data have been collected on the number of commercial burglaries and the time spent on preventive patrol. The time periods for which preventive patrol falls into each of the k categories are designated and the numbers of commercial burglaries associated with these time periods assigned to each of the k cells or categories. For each of these cells, we compute the mean number of burglaries. If in cell i there are n_i time periods then the sample mean x $K_i = \sum_{j=1}^{n_i} x_{ij}/n_i$ where x_{ij} is the number of commercial burglaries in time period j with preventive patrol i.

If preventive patrol has no influence on the number of commercial burglaries (successful) then the means in each category should be essentially the same. Thus, we formulate the so-called "null hypothesis" which can be written $H_o = \mu_1 = \mu_2 = \ldots = \mu_k$ where μ_i is the true or population mean for the i-th patrol level. If our previous arguments have any merit, we would expect the null hypothesis to be rejected. However, the μ_i are not directly measurable, we have only the sample means, K_i. The question of how close say K_1 and K_2 have to be before they are considered the same is one of selecting the "level of significance," if the sample size is given. If we choose a level of significance say of .05 then when we say that $K_1 = K_2$ at the 5% level of significance we mean that there is only a 5% probability that the two sample means would be this close if, in fact, preventive patrol does have an influence and the population means are different.

As these statistical techniques are discussed with reference to the analysis of police patrol problems, some common technical terminology must be employed. We cannot always repeat the standard textbook representations of terms like "level of significance." The reader is referred to standard textbooks on statistics.

The test is based on a comparison between the variance that occurs within a group, indicated by i, and the between group variance of group means. If the hypothesis holds, that the means of each group are the same, then the variance between groups should be small. The statistic used is the "F" statistic and is equal to the fraction s_m^2/s_p^2 where

s_m^2 is the between groups variance given by

$$s_m^2 = \left\{\sum_{i=1}^{k} (K_i^2/n_i) - K^2/N\right\}/(k-1)$$ and $N = \sum ni$ = total no. of observed commercial burglaries

s_p^2 is the total within-group variance given by

$$s_p^2 = \left\{\sum_{i}^{k}\sum_{j} N_{ij}^2 - \sum_{i=1}^{k} (K_i^2/n_i)\right\}/(\sum_i n_i - k)$$

A large value of F would indicate rejection of the hypothesis that the mean numbers of commercial burglaries were all the same regardless of the level of preventive patrol. The critical region (i.e., region for rejection) is $F > F_{(1-\alpha)}(k-1, \sum ni - k)$ where α is the chosen level of significance. For a given value of α, we know k-1 (one less than the number of patrol intensity categories) and $\sum ni$ is the total number of time periods for which data were collected on the beat or region in question. Thus, $F_{(1-\alpha)}(k-1, \sum ni-k)$ can be looked up in a standard table for the F statistic.

The assumption underlying this test is that the observations we've used are randomly selected from normally distributed populations of the same variance. In other words, we've assumed that the data collected when preventive patrol is at level i, i.e., the X_{ij}, are equivalent to random selections, for group i, from a normal population. We assume this for each of the k groups representing different levels of patrol technology. We further assume that the variance within each of these groups is the same. These assumptions can be checked if data are available. If they are valid, the test of significance using the F distribution as just described is known to be valid. In addition, "Investigation has shown that the results of the analysis are changed very little by moderate violations of the assumptions of normality and equal variance." *

Of course, if we do not reject our null hypothesis that preventive patrol doesn't influence the mean number of commercial burglaries then our theoretically based anticipation is disappointed. On the other hand, if our tests indicate rejection of the null hypothesis we may say with some confidence that preventive patrol efforts as classified for this test do influence commercial burglaries.

In the latter case, we could further test for differences in the individual group means, μ_i and μ_j, using methods similar to those described above.** This might yield some information about the nature of the differences among the means from which an indication of the effect of different types of intensity of preventive patrol might be obtained. Of course, we would expect that the means would systematically decrease as we move from group to group since the intensity of patrol increases.

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*Dixon, W. and F. Massey, Introduction to Statistical Analysis, (New York, McGraw-Hill, 1957) p. 151.

**ibid, p. 152-156.

The most arbitrary feature of this test is the classification of different patrol types and intensity into the k categories. Of course, the classification cannot be developed without a perusal of the data. Even then, we might want to try several different sets of categories. In any case, the sensitivity of the results to the category definitions should be investigated.

If data were available, we could perform an identical test substituting the number of on-view arrests for the number of commercial burglaries.

Cross-correlation

If we consider N_{ij} the number of commercial burglaries during time period i and T_i the time spent on preventive patrol during time period i as random variables, then a cross-correlation coefficient, ρ, can be calculated and used in a test of the dependence between the two random variables. If n is the number of time periods during which data were collected, then

$\bar{N} = \sum_i N_i/n$ and $\bar{T} = \sum_i T_i/n$. Now we can write:

$$\rho = \sum_i (N_i - \bar{N})(T_i - \bar{T}) \Big/ \left[\sum (N_i - \bar{N})^2 \quad (T_i - \bar{T})^2\right]^{\frac{1}{2}} .$$

Since the number of burglaries committed is usefully treated as a random variable and the time spent on preventive patrol, being a residual caused by random calls for service, can be treated as a random variable, this approach is reasonable. If ρ is small, the random variables T_i and N_i are relatively independent and increasing T_i could not be expected to influence N_i. On the other hand if ρ is large in absolute value, then the conclusion is that T_i and N_i are related. (We would expect ρ to be different from zero and negative.)

Tables exist[+] which give percentiles of the distribution of ρ assuming the random variables are independent. Thus, independence can be evaluated at some pre-determined level of significance.

[+] ibid, p. 468.

Aggregated regression tests

Although regression is the most common form of statistical test for the relation between two or more variables, the data related to a single area and reasonable time periods make it unlikely that they would lend themselves to regression analysis. The range of variation in the dependent variable, the number of commercial burglaries or the number of burglary arrests, is too small. However, it might be informative to run a regression on aggregated data. Still restricting attention to a single pre-determined area, we add the results of our data collection to develop weekly totals. Now, N_i' will represent the number of commercial burglaries during the study hours selected for the i-th week. Similarly T_i' is the preventive patrol time during the study hours during the i-th week. A regression of the form:

$$N_i' = \beta_o + \beta_1 T_i' + e_i$$

could be run to determine values for β_o and β_1. Then the hypothesis that $\beta_1 = 0$ could be tested in the usual way. Rejection of this hypothesis is an indication that weekly totals of burglaries and time on preventive patrol are related and the sign of β_1 (presumably <0) indicates the direction of the relationship. If $\beta_1<0$ and $\beta_o>0$ then β_o can be taken as a rough estimate of the upper bound on commercial burglaries. This last interpretation is risky since it calls for the linear extrapolation of the results into an uncertain region near the point where $T_i' = 0$.

It is quite likely that the results of the regression analysis will be disappointing in that the hypothesis $\beta_1 = 0$ will not be rejected. Again, this could be due to the predicted lack of variation in the weekly totals in N_i' and T_i'.

Conclusion of Descriptive Section on Patrol

The purpose of this phase of the study is to investigate two potential effects of police patrol activity. They can be summarized in the following questions: (i) Does the amount of time spent on preventive patrol on a beat in time period t have any influence on the number of commercial burglaries on the beat in some later time period? (ii) Does the amount of time spent on preventive patrol have any influence on the ratio of unsuccessful commercial burglaries to total commercial burglaries on the beat during the same time period?

Ideally, the procedure used to answer these questions would involve first the exact specification of patrol technology (e.g., 2-man squad cars patrolling the streets and alleys of a beat in a specified systematic manner). Then the intensity of the preventive patrol would be varied in a systematic pre-determined way and the value of the two independent variables measured. The specification of the patrol technology might be the product of an operations research analysis similar to that discussed in the next section of this report.

Our approach recognizes the cost of experimentation, both direct and indirect, in a real world environment as fraught with danger as the arena of police activity. We also recognize that our approach will be less accurate than direct experimentation. We further recognize that many police departments treat preventive patrol as a residual activity to be performed when a patrol unit is not engaged in answering calls for service. In these instances, no direct decision on preventive patrol is made by the police officials. This may or may not be an efficient patrol technology but it does seem to be a common one. Commercial burglary calls for service make up only a small fraction of the calls for service assigned to any particular patrol unit. Thus the amount of time, and when it comes during a watch, spent on preventive patrol is not determined by the rate of commercial burglary under this patrol technology. Apparently, the only way to increase the time spent on preventive patrol in this technology is to add additional patrol units.

Additional patrol units should result in a decrease in response time and an increase in the probability of space-time coincidence. Whenever a preventive patrol unit is able to observe a burglary in progress, it has achieved space-time coincidence. Both of these effects of increased preventive patrol activity should decrease the probability that a successful burglary can be committed.

If the decrease in the probability of success is made known, this should serve to convince some potential burglars that higher expected rewards await them in some other line of work. It must be stressed that increased effectiveness on the part of the police can be further enhanced by publicizing the resulting reduced probability of success. It is the probability of success, as seen by the potential burglar, that must be reduced by police activities if they are to have a deterrent effect. It might be just as effective and less expensive to lower this probability by introducing new technology which is not fully understood by the potential burglar but about which claims of effectiveness can be made. An example of such a device might be closed circuit television surveillance.

In this study we explicitly assume that no such devices are introduced over the suggested data collecting period. We also assume no information gap. The burglar knows the true ratio of unsuccessful burglaries to burglaries. However, he is not assumed to learn it immediately which accounts for the time lag in the deterrent effect.

There is serious doubt in some minds that police preventive patrol is really a deterrent to potential criminals. Clearly, if there were no police patrol activities at all, large numbers of the good citizens of an urban area would undoubtedly turn to crime. Evidence of such behavior can be found from descriptions of riot areas and of police strikes. On the other hand, rather large increases in police patrols in selected high crime areas often does not seem to reduce crime rates significantly. Perhaps it is safe to conclude that the presence of a police force does have a deterrent effect but this effect may be relatively insensitive to rather large changes in police patrol inputs. If this is the case, the statistical analysis suggested above will reveal very little influence for preventive patrol times on the number of burglaries committed. The range of variation in preventive patrol time likely to be observed may simply not be great enough.

Final Comments

This section of the report has been written without the aid of analytical expressions and with a minimum of technical language. It is impossible to describe fully the details of the analytical procedures employed in this form. However, the often stated reluctance of police officials to read technical material prompted us to choose it.

The material discussed thus far in the report has been covered in the quarterly reports. This fact and the nature of the material permitted a rather non-technical presentation of the work. The next sections deal with the use of operations research techniques, particularly search theory, and the use of analytical expressions and technical language cannot be completely avoided.

PART II: NORMATIVE MODELS

Preventive Patrol Model - 1

There are many reasons for developing a preventive patrol model and a preventive patrol assignment algorithm. The overriding reason, however, is the current lack of a validated model and assignment method. Several authors have proposed patrol models and carried them to various stages of completion. Elliott [1] has used a form of Koopman's [2] search model to justify the increased use of patrol forces against the crime of burglary in Syracuse, New York. He has also done some experimental work to determine the ability of police officers in patrol cars to spot breaks in windows or doors. Larson and Blumstein [3] used their version of a Koopman-type search model to demonstrate the low probability of detecting crimes-in-progress by a patrol officer. Olson [4],[5] used the Blumstein-Larson model with robbery data in Chicago to allocate patrol units to specific sectors and to compute a theoretical upper bound on their probability of detecting at least one crime-in-progress. This assignment method was not subjected to a street test. None of these models or assignment methods have been fully exploited, or extended in a manner to permit the generation of actual patrol routes for police units. A discussion of the reasons for conducting more work in this area follows:

Currently, police departments believe that preventive patrol is effective as a deterrent and as a means of suppressing crime. When crime rates rise, the departments request funds for "more men on the street." The departments have no quantitative methods for justifying this manpower increase. An attempt to develop a statistical relationship between patrol input and output is discussed in the previous section. Very few departments even maintain the data to support the positions that street patrol either deters crime, or acts as an effective agent in detecting crimes-in-progress and in apprehending the criminals. The budgets for police departments usually expend over 90% for personnel costs in wages and benefits. This means that the allocation of manpower has the overriding effect on the quality of police service per dollar that a community receives. A quantitative method of assessing the effects of preventive patrol is therefore necessary to improve manpower allocation within a department.

The current level of allocation methods seldom goes beyond pin maps that show the occurrences of crimes in the city. It is assumed that police officers viewing these maps will then conduct effective preventive patrol. Some cities, such as St. Louis, use a computer to generate crime density maps. These are more useful since a copy can be retained by a patrol officer in his car. This still does not solve the problem of distributing the police patrol effort in the most effective manner. Quite often police departments will allocate their available patrol effort in a direct proportion to the level of criminal activity. The search models in this paper and those of Elliott and Larson show that this is not the most effective allocation of effort.

Two problems must be faced by a police administrator: (1) how can I best conduct preventive patrol with my existing resources, and (2) how can I best improve the effectiveness of my preventive patrol effort. These problems imply that an assignment method should be able to use estimates of current patrol performance for the first problem, and to encompass enough of the pertinent patrol crime, and environmental characteristics to assist the police in improving their performance through training programs or by physically changing some of the model parameters.

This second problem leads to another reason for developing an analytical patrol model. Any model will become more effective as it considers more of the factors bearing on the patrol problem. In adapting the model to a particular city (or area within a city) police officers should work with the model builders. When they are asked to estimate or to measure the parameters included in the analytical model, they will probably become much more aware of the total preventive patrol problem. This model can serve as a talking point for the planning personnel and the operations people.

In many metropolitan areas police officers do not work a particular beat for extended periods of time. Sick leave, court time, vacations and transfers require that many different police officers patrol a given area. As a result, many officers do not have an intimate knowledge of the existing crime patterns based on their experience in an area. An assignment method using the latest crime data and allocating the officer's patrol time within the area could provide more effective coverage than an officer acting on his own.

Police often regard areas with a lot of bars and other areas of pedestrian congestion as potential areas for disturbances or disorderly conduct. As a result, they often concentrate patrol effort on these streets. Quite frequently, the street crimes of robbery and burglary do not occur in these same areas. A patrol model would force the police to select particular crimes for preventive patrol, or to weight the different crimes according to the emphasis that they wish to place upon them. The model would then allocate an officer's patrol time accordingly.

Finally, very few humans can act in a truly random manner. Preventive patrol should be random, or a criminal would simply plan his crime for a time when no patrol units were in the vicinity. The allocation model should incorporate a feature that will generate a random patrol path for the police officer. By random, we mean a path that cannot be predicted on the basis of past history.

The models presented here will review some of the past work and extend it. Finally a model differing from the earlier models will be proposed for future implementation. It is a Markov decision process that provides a method for generating random patrol paths and also incorporates some of the features of the earlier patrol models.

Almost every police department performs some preventive patrol. The amount and the method vary considerably. For general application, assignment models should consider the nature of these methods.

Many departments rely solely upon the free time of their beat patrol officers for preventive patrol. The amount of preventive patrol time depends upon the number of calls for service that they are assigned, the length of time spent on these calls, the amount of time spent on administrative duties, lunch, and other duties such as parking tickets or traffic that remove patrolmen from patrol status. In many departments, very little preventive patrol is done. In those departments which try to have a very short response time to a citizen call, however, the individual officer cannot be out of patrol status for much more than 60% of the time. This occurs because a short response time is guaranteed only if there is a high probability that a nearby officer is not busy at the time of the call. Thus, an effective response force and effective preventive patrol are complementary. Departments that have the ability to respond quickly to a call then have beat patrolmen with a fair amount of time available for preventive patrol -- at least 25-30%. Some departments try to assign beat cars so that each car has nearly 50% of his time available for patrol.

Any patrol allocation for beat patrol units should be done on the basis of their expected patrol time -- not their full shift. One allocation method is based on queueing theory. If K is the number of patrol cars, $^1/\lambda$ is the average time required for a patrol unit to service a call, and μ is the average rate of incoming calls for service to patrol units, then the probability that a random patrol car is patrolling is $\rho = 1 - \frac{\mu}{K\lambda}$. The expected amount of preventive patrol is ρK, which is the expected number of units in patrol status. Additional times such as lunch periods could also be considered in estimating the availability of preventive patrol units.

Once the amount of preventive patrol time is estimated, the allocation model will indicate the amount of preventive patrol time to be spent by the beat officer in different sections of his beat. Because calls for service or other interruptions (stopping a traffic law violator) occur randomly during his shift, the allocation model cannot assign definite times for his preventive patrol.

Some cities, such as St. Louis and Chicago, have special units that operate only as preventive patrol units. While these units would respond to emergency calls, they do not receive normal calls-for-service from a dispatcher. These units often operate within high crime areas of the city on the evening watch -- 6 P.M. to 2 A.M. Even though they have no call answering duties, these units often conduct a large number of street stops that remove them from patrol status. For a 21 day period in Chicago, the overall average of availability for one preventive patrol group was 67.3%. This demonstrates that the operation of even a full time preventive patrol unit must be examined before an assignment algorithm is prepared. Units that did not perform as many street stops of pedestrians and vehicles would have more time in patrol status.

With either the preventive patrol efforts of a beat officer or a special force, police administrators often allocate their men by considering the major crime problem in an area and telling their men to look out for it. Sometimes the area chosen for a particular mission is restricted in size -- perhaps a public housing neighborhood -- or much larger. Aside from a pin-map type of crime analysis no quantitative justification is given for the amount of patrol effort directed towards these areas.

The model used in [4,5] selected areas within the city that had the highest incidences of robbery, estimated the effectiveness of police patrol in the area by a random search model, and assigned the number of units to each area with the objective of maximizing the space-time coincidence of at least one patrol unit with a robbery subject to a constraint on the number of available patrol units. Space-time coincidence simply refers to a case where a patrol vehicle passes a point at the same time that an observable crime is taking place. This does not insure that the police will detect the crime or apprehend the criminal. The probability of space-time coincidence is then an upper bound on the probability of detection. This allocation model was not subjected to an actual street experiment.

Larson [6] provided a more detailed allocation method similar to the above, except that it intended to allocate patrol effort to individual streets and alleys. The benefits and restrictions on this method of allocation will be discussed later.

Finally, there are models for allocating the location and tactics of all available patrol units when a crime has just been reported, and a description of the criminal or his vehicle is known. This is the "hot-pursuit" case and it will not be treated in this paper. Bottoms [7] discussed this and proposed methods for locating the trapping forces. The methods are undergoing study and experiment with the Washington, D.C. police department at this time.

Up to this point we have discussed some reasons for seeking a quantitative method of preventive patrol allocation, and we have given qualitative descriptions of some methods for allocating patrol effort. Now, some of the factors influencing the selection of areas for patrol and the assignment of patrol effort to these areas will be discussed.

The first characteristic about an area is the type of crime problem within the area. As mentioned before, preventive patrol can be considered effective only against crimes which can be recognized by police on patrol. Police patrol is normally confined to public areas. Crimes such as robbery, purse snatching, and burglary (where signs of forcible entry or the transport of stolen goods are evident to a patrolling unit) offer some possibility of detection by preventive patrol. While preventive patrol is supposed to deter crime -- not just capture criminals in the act -- it can be argued that no criminal is deterred unless he feels vulnerable to detection and apprehension. It seems questionable that preventive patrol has any influence on crimes that are not in theory detectible by them. For the most part, murder, rape, and serious assault have in the past been committed on private property out of view of patrol units by persons known to the victim. Hence, police have regarded these crimes as "non-preventable" by patrol. Recently, some crime analysis has shown that murder as an outgrowth of another felony such as robbery or burglary is on the increase. As a result, a greater number of murders are committed in outdoor locations where patrol could be effective. As an example, these figures from the Chicago Police Annual Report 1969 are given: [8]

Table 1

	1968	Percent of Total	1969	Percent of Total
Total Murders	647		715	
Robbery, Burglary, etc. Motivation	65	10.0%	102	14.3%
Street, Alley, Park, Open Lot Location	212	32.8%	258	36.1%
Outdoor Residence Area	19	2.9%	31	4.3%

These figures are not given to suggest that murder is now a prime candidate for suppression by preventive patrol. Rather, they are given to indicate the type of analysis that must be done within a department to characterize their crime problem. This data shows that over one-third of the murders occur in areas available to police patrol, and that there may be a trend for this type of murder to increase. Further analysis could show that these types of murder occur in a relatively small section of the city.

Similarly, an analysis of burglaries and robberies within certain neighborhoods could show them occurring out-of-view of any possible patrol unit. This would be particularly true of daytime burglaries in multi-story apartment buildings where the burglar does not have to make a forcible entry on the ground floor, and when he confines his attention to cash or easily concealed items.

Once this crime analysis is complete, an estimate of the number of "viewable" or "preventable" crimes occurring in a region can be made. The purpose of the first part of this research project was to predict the number of commercial burglaries in a neighborhood. The estimate or prediction of the number of crimes committed in a particular area can be done in many ways -- including the use of past history. The predicted number of crimes in an area for a given time period can then be divided by the total number of crimes predicted for the entire city in that time period. This gives a relative frequency of crime occurrence in an area that can be used as an estimate of the probability of a crime occurring in that area. This is the first and most important crime factor for the allocation model -- the probability by time period (perhaps an 8 hour shift) and area of the occurrence of a viewable crime.

The second factor to consider is the relative seriousness of each type of viewable crime. Victims would generally prefer to have their homes or businesses burglarized as opposed to themselves being robbed and subjected to the risk of bodily injury. Any system that weights the seriousness of crime types relative to each other will be largely subjective. One of the most elaborate surveys that attempted to determine relative crime weights was conducted by Sellin and Wolfgang [9]. The persons who participated in the survey were college students, police officers, and judges. The purpose of the weights was to demonstrate the relative seriousness that this group attached to particular crimes committed by juveniles. One would not be too surprised if these weights did not correspond to the relative seriousness that residents of high crime areas would give to crimes in their neighborhoods.

Until surveys are conducted within the neighborhoods of a city, the Sellin and Wolfgang weights are probably as good as any. As an example, some weights from their survey are given:

Homicide	26
Rape	12
Robbery	7
Aggravated Assault	7
Burglary	3
Theft ($50 or over)	3
Auto Theft	2

If only one crime type is being considered, such as commercial burglary, the relative seriousness could be judged by using the value of the stolen goods. Police allocation to some areas could be affected by considering the potential loss or damage if burglaries or robberies occur there.

So far, we have discussed methods of estimating the probability of a crime occurring and the relative seriousness that potential victims accord these crimes. In addition to these factors, a patrol allocation model should consider its effectiveness in detecting particular crimes in an area, and any particular costs or risks associated with this process. Taken together, these factors will then determine the areas for search and the assignment of manpower to them.

As inferred earlier, the mathematical allocation models will use the maximization of the probability of space-time coincidence as an objective function. This probability would become the probability of detection if the conditional probability of detection given space-time coincidence were known. When empirical data on the number of detections is used in an allocation model, this distinction between space-time coincidence and detection is unnecessary. If empirical data is unavailable better estimates of police performance can still be made by considering environmental factors such as lighting or distance that can affect the detection of a criminal event. An examination of the particular streets and alleys could provide a relative measure of difficulty in observing a criminal event by a patrol unit. This relative measure could then be used in the allocation procedure.

Differences in the distance from a squad car to building doorways would affect the ability of an officer to see signs of forcible entry. This is a major difference in patrolling residential areas as opposed to commercial areas. Similarly, the lighting in streets, alleys, or in commercial establishments after business hours varies greatly. Finally, the amount of time that a criminal event is viewable as a crime to patrol forces will affect the probability of space-time coincidence -- and then detection. Each crime type has a characteristic time associated with it. A street robbery may take less than a minute while a commercial burglary where goods other than cash are taken might take 30 minutes to complete. The model builder can bring the police into the problem by asking for their estimates of these time values. The discussion will inform the police about the nature of the allocation model and, undoubtedly, receive their criticisms and suggestions.

In a similar manner, the model builders should determine if certain areas place a higher cost on patrol. This additional cost could result from restrictions on the patrol speed, a requirement for additional manpower or equipment in the area, or a subjective estimate of the relative risk of police activity in that area as opposed to others.

These factors can be included in some of the mathematical allocation models. The description of the models will show how they can be included, and how they might affect the final result. It will also be clear that some of these factors, such as the subjective weightings on crime types, difficulty of detection, and the risks or additional cost of patrol can be omitted from the model.

Preventive Patrol Model - 2

Consider a one-car (preventive patrol) sector (beat) consisting of N streets, numbered 1, 2, ..., N. Assume that the preventive patrol policy for this sector prohibits U-turns. This restriction is easily modified for any street in the models presented in this paper. Since U-turns are not allowed, we require a notation which specifies the possible location of the patrol car as well as the direction in which it is patrolling. For this purpose let the pair of integers 2i-1 and 2i represent the two directions of patrol for street i(=1,..., N). Finally set M = 2N and let the integer 0 denote the location of the patrol car prior to commencing patrol (e.g., garage, police station, etc.). Street 0 could also represent the street or set of streets the patrol car must travel in order to reach the sector at the beginning of a shift. Figure 1 illustrates such a street-direction sector network.

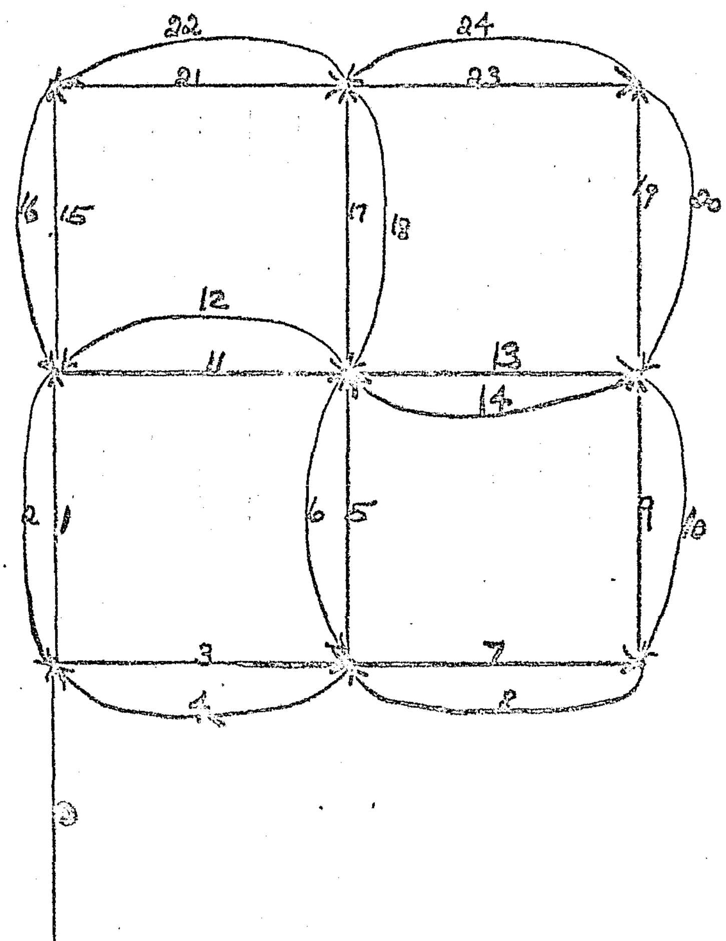

Figure 1 N=12 Street Sector

One of the major objectives in preventive patrol is that the patrol schedule (policy) be non-predictable. That is, there should be no way in which a potential criminal can better predict when the next patrol car will visit a given location knowing the times of the previous visits to this location by the patrol car than not knowing these times mathematically, let $T_1^j, T_2^j, \ldots, T_n^j$ be the times of visits 1, 2, ..., n to location j in the sector by the patrol car and set $Y_{n+1}^j = T_{n+1}^j - T_n^j$. T_{n+1}^j is the time of the (n+1)st visit to location j and Y_{n+1}^j is the (elapsed) time between the nth and (n+1)st visit to location j. A patrol policy is said to be "random" or non-predictable in the above sense if

$$Pr\left\{Y_{n+1}^j = \tau \mid T_1^j, \ldots, T_n^j\right\} \tag{1}$$

$$= P_r\left\{ Y_{n+1}^j = \tau \right\}$$

for all locations j in the sector, for all n = 0, 1, .., and all time periods $\tau = 1, 2, \ldots.$

(1) in words says that the unconditional probability that a patrol car visits location j τ time units in the future for the (n+1)st time is equal to the (conditional) probability of this same event occurring given the observed times of the previous n visits to this location by the patrol car. If statement (1) holds for all locations j in the sector, then we say that the patrol schedule is random.

The use of random patrol schedules in preventive patrolling is partially based on the premise that it will discourage potential criminals who find that they cannot predict the arrival patterns (times of visits to streets) of patrol cars. It is this property, randomness of patrol, that will be maintained in the models presented in this paper. Next, let X_τ denote the location (street and direction of patrol) of the patrol car at time τ (=0, 1, ...). X_τ takes values in the set $S = \{0, 1, \ldots, M\}$. A random patrol schedule has the property that for any sequence of locations the patrol car has visited up to time τ, say $0, i_1, i_2, \ldots, i_\tau$, the probability that the next location patrolled is j conditioned on $0, i_1, \ldots, i_\tau$ is only dependent on the last location patrolled, i_τ. This statement must hold for any sequence of locations $0, i_1, i_2, \ldots, i_\tau$ and j in the sector specified by S, and any time period τ (=0, 1, 2, ...,). Mathematically, this statement, which is equivalent to (1) is given by saying that the

$$\Pr\left\{ X_{\tau+1} = j \mid X_0 = 0, X_1 = i_1, \ldots, i_\tau \right\} = \Pr\left\{ X_{\tau+1} = j \right\} \qquad (2)$$

for all $\tau = 0, 1, \ldots,$ and any sequence $(0, i_1, i_2, \ldots, i_\tau)$ where the i_μ's are in S.

Most of the models appearing in the literature on preventive patrol, as described earlier in this report, determine either the optimal number of visits to each location in a sector, "optimal coverage rates", assuming that patrol is random, or assumes coverage rates are given and then determines random patrol schedules. For examples of both cases see Larson [6] and Rosenshine [10]. Given crime statistics for a sector during a specified interval of time and other statistics such as the speed of patrol, the criterion most often used in models for determining optimal coverage rates is to maximize the total probability of space-time coincidence or to maximize the expected number of detections. The latter criterion in some instances is equivalent to the former.

The models presented in this paper combine these two decision procedures. The allocation model (determining optimal coverage rates) and the random patrol schedule model, are combined into a single model which meets the desired criteria. Also, the models presented in this paper have the added advantage that the optimal patrol schedules can be computed by mathematical programming using readily available computer programs.

The output of the two models is an optimal matrix, called P^o, of transition probabilities. That is,

$$P^o = \begin{pmatrix} p^o_{00} & p^o_{01} & \cdots & p^o_{0M} \\ p^o_{10} & p^o_{11} & \cdots & p^o_{1M} \\ \vdots & & & \vdots \\ p^o_{M0} & p^o_{M1} & \cdots & p^o_{MM} \end{pmatrix}$$

where the numbers (transition probabilities) p^o_{ij} in P^o have the following interpretation.

Let i and j be any two locations in the set S and let τ (= 0, 1, ...) be any time period. Then

$$p^{o}_{ij} = \Pr\left\{ X_{\tau+1} = j \mid X_{\tau} = i \right\}$$

= probability (conditional) that given location i is now being patrolled (time τ) the next location patrolled is j (in time $\tau + 1$).

The probabilities in P^{o} will be "close to" zero for pairs of locations, say (i,j), where j cannot be patrolled next upon the completion of patrolling i without either making a U-turn or passing through an intermediate location. More will be said on this point later in the paper.

Next, using the matrix P^{o}, random patrol schedules can be determined by the use of Monte-Carlo (simulation) techniques. The resulting patrol schedule or schedules give for any location of the patrol car during any time period, the next location to be patrolled. The user would then generate a series of schedules (from the Monte-Carlo model) which would be used, say on successive tours during the same shift.

Finally, the patrol schedules generated would be simple in form. In most cases, for each shift and patrol sector, a patrol schedule would consist of two columns of basic information as illustrated in the following example.

Patrol Schedule

(Sector No., Shift No.)

Present Location (Street and direction of patrol)	Next Location to Patrol (Street and direction of patrol)
0	i_0
1	i_k
.	.
.	.
.	.
M	i_M

Note that in the schedule, if the patrol car has finished patrolling location i in S, then the next location to patrol is location j_i in S.

In Appendix B, a preliminary description of the preventive patrol decision process is presented. Derman's approach [11] is used in the construction of the patrol decision process as well as in the statement of the models.

APPENDIX A

The method of allocating police to areas in the city to combat street crime follows an early model of Blumstein and Larson [3]. In this model

T_c = the time that a criminal event is viewable by patrol units

B = the number of miles of streets and alleys for patrol per square mile

A = the number of square miles in the patrol area

S = the speed of patrol

K = the total number of police patrol units

P_{st} = the probability of space-time coincidence

and

$$P_{st} = 1 - \exp\left(- \frac{KST_c}{BA} \right)$$

or

$$P_{st} = 1 - \exp\left(- \rho\frac{KST_c}{BA} \right)$$

when we take into account the probability ρ of a patrol unit being in patrol status. The complete description of the development of this model is given in [3] and [4]. The probability of space-time coincidence is the probability of at least one patrol unit of the K units in area A achieving space-time coincidence with a given criminal event. This model assumes that the K units all patrol region A in a random manner. In other words, one unit does not patrol a region of $\frac{A}{K}$, but each unit patrols the same area A.

This probability, P_{st}, was used in the Koopman search model which is written as

$$\text{Max } P[\varphi] \quad \int_{-\infty}^{\infty} p(x)\left[1 - \exp\left(-\varphi(x)\right)\right] dx \qquad (1)$$

$$\text{subject to } \int_{-\infty}^{\infty} \varphi(x)\, dx \leq \alpha \qquad (2)$$

$$\varphi(x) \geq 0 \qquad (3)$$

$$p(x) \geq 0 \qquad (4)$$

where p(x) is the probability of a crime occurring between region x and x+dx. $\varphi(x)$ is a measure of the effort expended to achieve space-time coincidence in the region x and x+dx with the criminal event, and α is the resource constraint on the total amount of effort available. The expression $1 - \exp(-\varphi(x))$ in the integral of equation (1) is the probability of space-time coincidence given that a criminal event occurs. In the area allocation model

$$P_{st} = 1 - \exp\left[-\rho\frac{KST_c}{BA}\right] = 1 - \exp\left(-\varphi(x)\right) \qquad (5)$$

$$\text{or} \qquad \varphi(x) = \rho\frac{K(x)ST_c}{BA} \qquad (6)$$

$$\alpha = \rho\frac{KST_c}{BA} \qquad (7)$$

In this case we collapse the sectors of search onto the real line, x, since the search area is characterized by the total miles of streets and alleys for patrol in an area. The parameters S, T_c, B, A, and ρ are considered fixed for the allocation problem. The variable controlled by the decision maker in this model is K(x), or the amount of the total number of patrol effort expended between x and x+dx. For example, if the value of K were 20 patrol units and the value of K(x) that maximized equation (1) for the area represented between x_1 and x_1+dx were .6, then

$$.6 \times 20 \times 8 = 96 \text{ hours}$$

of the maximum available preventive patrol time -- or 12 patrol units -- would be allocated between x_1 and x_1+dx. The actual amount of preventive patrol time would be ρ96 hours.

This model could include other factors besides the estimate of the probability of a crime, the time duration of the criminal event, and the patrol unit availability. Let $0 \leq q(x) \leq 1$ be the weighting factor on the relative difficulty of detecting a criminal event given space-time coincidence. If known, this could also be the probability of detection given space-time coincidence in the area represented by the interval x to x+dx. Similarly, w(x) would be a relative weighting factor with $w(x) \geq 0$ that gives an indication of an additional strain on resources for patrol between x and x+dx. Finally, the factor $v(x) \geq 0$ could represent the relative value of detecting a criminal event between x and x+dx. If more than one type of crime is being considered, v(x) could be a composite of Sellin-Wolfgang crime weights to indicate the relative seriousness of crimes occurring between x and x+dx.

The allocation model would then be written as

$$\text{Max } P[\varphi] = \int_{-\infty}^{\infty} v(x)p(x)[1 - \exp(-\rho q(x)\varphi(x))]dx \qquad (8)$$

$$\text{subject to } \int_{-\infty}^{\infty} w(x)\varphi(x) \geq \alpha \qquad (9)$$

$$\varphi(x) \geq 0,\ v(x) \geq 0,\ w(x) \geq 0,\ q(x) \geq 0,\ p(x) \geq 0$$

which can be solved in a similar manner to equation (1).

This model is usually described in terms of one crime type which has a characteristic value of T_c. In some areas, only one type of street crime is a major problem. The model can be very simply applied as described. This restriction can be removed, however. Consider three areas for patrol:

$$A_1 \qquad A_2 \qquad A_3$$

where the number of robberies, burglaries, and acts of vandalism viewable by patrol units are estimated as

$$N_1^R,\ N_1^B,\ N_1^V,\ ---,\ N_3^R,\ N_3^B,\ N_3^V$$

respectively. Assume relative crime seriousness weights of 0.7, 0.3 and 0.2, respectively. Let the time duration that these crimes are viewable be T^R, T^B and T^V.

The estimated probability of robbery in area one given that a robbery occurs is $p_1^R = \frac{N_1^R}{N_T^R}$ where N_T^R is the total number of robberies $N_1^R + N_2^R + N_3^R$. Similarly, the values of $p_2^R, \ldots, p_3^V$, etc. are found. The expected time duration of a viewable street crime in area one is

$$T_1 = p_1^R T^R + p_1^B T^B + p_1^V T^V \tag{10}$$

and the crime seriousness factor is

$$v_1 = \frac{0.7N_1^R + 0.3N_1^B + 0.2N_1^V}{0.7N_T^R + 0.3N_T^B + 0.2N_T^V} \tag{11}$$

The probability of a crime occurring in area one is then

$$p_1 = \frac{N_1^R + N_1^B + N_1^V}{N_T^R + N_T^B + N_T^V} \tag{12}$$

Using the discrete form of the integral equation, we have

$$\text{Max } P[\varphi_i] = \sum_{i=1}^{3} v_i p_i \left[1 - \exp(-q_i\varphi_i)\right] \tag{13}$$

subject to

$$\sum_{1}^{3} w_i\varphi_i \leq \alpha \tag{14}$$

with

$$\varphi_i = \rho\frac{ST_iK_i}{BA_1} \tag{15}$$

Larson [6] later developed a model which sought to allocate the relative amount of patrol effort to each block of a street or alley. The function that specifies the amount of coverage that one street or alley receives relative to another is $0 \leq e(x) \leq 1$. A value of $e(x) = 1$ indicates the maximum amount of coverage. The total distance patrolled in a sector sweep is

$$L = \int_o^D e(x)\ dx \tag{16}$$

where D is the total number of miles of streets and alleys in the patrol sector. $P_n(x_0, \ell)$ is defined as the probability that at least one patrol segment overlaps a point x_o in a patrol of length ℓ given that the patrol is divided into n equal length segments. Larson obtains this probability as

$$P_n\ (x_o,\ \ell) = 1 - (1 - \frac{\ell}{n}\ \frac{e(x_o)}{L})^n \tag{17}$$

which becomes

$$P_n\ (x_o,\ \ell) = 1 - \exp\ (- \frac{\ell\ e(x_o)}{L}) \tag{18}$$

as n becomes large. If we call the speed of patrol s, the effective sector sweep time is

$$T = L/s$$

and

$$P_n\ (x_o,\ t) = 1 - \exp\ (- \frac{te(x_o)}{T}) \tag{19}$$

where $t = \ell/s$ if $P_n(x_o,\ell) = P_n(x_o,t)$ is the time duration of the patrol. This time duration of patrol is the time duration of the criminal event, not the total amount of time a unit spends in patrol. This is because we are interested in the probability of the patrol unit passing point x_o while the crime is in progress.

This last equation is interpreted as meaning that the passage of point x_o is a Poisson process with rate $\frac{e(x_o)}{T}$. The probability density function for the time between passings of x_o has a negative exponential distribution with mean $T/e(x_o)$. The patrol rate at point x_o is

$$v(x_o) = \frac{e(x_o)}{T} \tag{20}$$

which is the average number of patrol passes of point x_o per unit time. Street segments between adjacent intersections are considered the smallest possible patrol segment. If one point of this segment is passed, all points on that street segment are passed.

Before proceeding, the reader may wish to compare Larson's result in equation (19) to Elliott's which is

$$p = 1 - \exp(-t/T) \tag{21}$$

where Elliot calls t the amount of time that the crime is viewable and T as the average time to patrol all the miles of streets and alleys in a patrol sector. Larson states that this interpretation of T is incorrect. Rather T should be interpreted as the average time between passings of a point of maximum patrol coverage.

If the patrol unit is performing other duties, such as answering calls for service, for some fraction $(1-\rho)$ of the total patrol time, the average number of patrol passings of point x_o per unit time becomes

$$v(x_o) = \frac{e(x_o)}{T} \rho \qquad (22)$$

Using the same weighting factor for the difficulty of observing a crime in progress at x_o as was done in equation (8), the last equation could be written as

$$v(x_o) = \frac{e(x_o)}{T} (\rho) \; q(x_o) \qquad (23)$$

This would now have the effect of altering the meaning of the probability to a probability of detection instead of space-time coincidence.

This allocation model is still in Koopman's form of

$$\text{Max } P = \int_{-\infty}^{\infty} p(x) \left[1 - \exp (-v (x) t \right] dx \qquad (24)$$

subject to

$$\int_{-\infty}^{\infty} \varphi (x) \; dx \leq \alpha, \; \varphi (x) \geq 0 \qquad (25)$$

where $v(x)t = \varphi(x)$. In this equation, Larson interprets α as a measure of the total amount of search available to achieve space-time coincidence with a criminal event of time duration t. In this case

$$\alpha = \rho \; s \; t \, K \qquad (26)$$

where α is measured as the total number of miles covered by the patrol units during a criminal event of duration t. Larson's equations can include the other weighting factors to be rewritten as

$$\text{Max } P = \int_{-\infty}^{\infty} v(x) \; p(x) \left[1 - \exp\left(\frac{-\rho \; q(x) e(x) ts}{L}\right) \right] dx \qquad (27)$$

$$\text{s.t.} \int_{-\infty}^{\infty} w (x) \; \varphi (x) \leq \alpha \, . \qquad (28)$$

There is a major problem in implementing Larson's model that also occurs to a lesser degree in the area allocation method used by Olson in [4]. In Larson's model, search effort is allocated to street and alley segments between adjacent intersections. Neither the objective function nor the constraints guarantee that search effort will be assigned in a manner to make the search by a patrol unit feasible in a physical sense. In other words, search effort could be allocated to streets and alleys that have no common intersections, so that a patrol car would have to cover streets or alleys which had no effort assigned in order to reach an assigned search segment.

Consider the street and alley search allocation as the assignment of flow rates between nodes (intersections) in a network. The flow rate is $\frac{e(x_o)}{T}$. In this problem, some of the flow rates would be impossible to achieve because there are either no connections between a flow source and a segment, or the connections have an insufficient capacity to achieve an assigned flow rate on an intermediate link. Therefore, the search effort allocated to the different links could only be used as constraint conditions in a mathematical program that tried to approximate the search allocation-assigned flow rates as closely as possible, and still achieved a physically realizable network flow pattern. Once the physically attainable, nearly optimal flow rates were found, a method similar to Rosenshine's [10] could be used to obtain the transition matrix of a Markov chain that results in achieving the expected flow rates. This transition matrix can then be used in a Monte Carlo program to generate random patrol paths for the search units which would tell the patrol unit which turn to make as it reached each intersection. By the end of the tour, the unit would have achieved a coverage of the network approaching the optimal allocation of search effort. Appendix B of this report will describe a Markovian decision process for allocating search effort that avoids this two-step (Larson-Rosenshine) process.

The generation of a random patrol will then be accomplished in a more straightforward manner. In the area allocation, a similar problem occurs when a fractional unit is assigned to a search sector. By multiplying the amount of time a unit has for patrol by this fraction, a unit could be assigned for that amount of time in the sector. The rest of the unit's patrol time would be spent in another sector. If there were a significant amount of time required to travel from one sector to another, this time must be deducted from the total amount of time that the unit spends on preventive patrol. Then the problem must be reworked with this new time.

The next section will discuss the methods of solution of the search problem with some interpretations of the results. The Larson allocation method could still be useful for other types of surveillance -- such as closed circuit TV -- or other types of protective devices that did not have to be transported from one segment to another. While his model will receive no further treatment, the methods of solution could be used with his model for these purposes. While the sector allocation method is not as precise as a street by street assignment of effort, it might be more practical to implement since patrol units may resent the specific driving instructions.

The first method for solving problems written as equation (1) was given by Koopman [2]. This method is graphical, and Larson [6] gives an example of the solution as applied to police patrol. As such, the method will not be described in this paper. If a department has a small allocation problem (such as a moderate size city with relatively few candidate sectors for patrol), this method of solution could be easily implemented since it does not require high mathematics or access to a digital computer.

The method of solution programmed by Olson [4] used a mathematical programming solution developed by Charnes and Cooper [11]. This method can be easily programmed for a digital computer. For this case, a discrete version of the integral equation is used and the problem is

$$\text{Min} \sum_{j=1}^{n} \exp(-\eta \varphi_j)\, p_j \tag{29}$$

subject to

$$\varphi_j \geq 0, \quad \sum_{j=1}^{n} \varphi_j = 1 \tag{30}$$

which is equivalent to the problem

$$\text{Max} \sum_{j=1}^{n} \left[1 - \exp(-\eta \varphi_j)\right] p_j \tag{31}$$

subject to equation (30).

In the discrete form, p_j is the a priori probability of the criminal event occurring in sector j and φ_j is a normalized parameter related to the effort allocated for search in sector j. The exponential quantity in equation (29) is the probability of not achieving space-time coincidence given a criminal event occurs in sector j where $\eta\varphi_j$ effort has been allocated. In the sector search model, the decision variable is the number of patrol units allocated to sector j. If the total number of patrol units available is U, then we have

$$\varphi_j = \frac{K_j}{U} \tag{32}$$

where K_j is the number of patrol units assigned to sector j. From the form of equation (32), we see that the constraint condition, equation (30), is satisfied. Recalling the form of equation (5), we have

$$\eta\,\varphi_j = \frac{\rho S T_c}{BA} K_j \tag{33}$$

so
$$\eta = \rho \frac{S T_c U}{BA} \tag{34}$$

for all search sectors j. The value of η in this equation is similar in interpretation as the quantity α in equation (26) for Larson's street allocation model. Namely, it is a measure of the total amount of resources available on the average (ρ is an average value) to achieve space-time coincidence with a criminal event.

The computer (or hand solution) is as follows. First rank the values of p_j with the highest value first. Then take the natural logarithms of the ranked values of p_j. Let $\hat{p}_j$ represent the natural logarithm of the *a priori* probability p_j. Let j = 1, 2, --, n now represent the order of the j-th sector in the ranking. If

$$\hat{p}_1 - \eta \geq \hat{p}_2 \tag{35}$$

then all of the search effort is allocated to the sector with the greatest *a priori* probability of a criminal event. If

$$\hat{p}_2 > \hat{p}_1 - \eta \tag{36}$$

then the second sector is added for search. The sectors receiving any search effort at all are found by iterating through the equation

$$\hat{p}_{n_J} > \frac{1}{n_J}\left(\sum_{i=1}^{n_J} \hat{p}_i - \eta\right) \geq \hat{p}_{n_{J+1}} \tag{37}$$

where J stands for the number corresponding to the last sector selected for search. Once the value of J is found in equation (37), the value of φ_j is found from

$$\varphi_r = \frac{1}{\eta}\left[\hat{p}_r - \frac{1}{n_j}\left(\sum_{j \in J} \hat{p}_j - \eta\right)\right] \tag{38}$$

where

$$r \in J.$$

This method of solution could be altered to solve Larson's street allocation problem as well. The more general case where weightings are used will be shown later.

Our review of the past models of police patrol and their methods of solution is now complete. While the methods described can solve the allocation problems, other methods using duality will be described in order to place this problem in a more general math programming format and to permit better economic interpretations of the results.

Several papers have been written that show the relationship of the search problem to the Neyman-Pearson problem. Wright and Francis [13] found a dual problem to the linear form of the Neyman-Pearson problem. Yen [14] used a theorem by Wagner [15] to write the Wolfe dual of the search problem, and Meeks [16] has obtained duality relationships for the non-linear Neyman-Pearson problem. Some of their results will be repeated here in the context of the preventive patrol problem.

Let a and b be real numbers or $\pm\infty$ with $a < b$. Let β and γ be functions on (a,b) for which

$$-\infty \leqq \beta(x) \leqq \gamma(x) \leqq \infty \qquad \text{for } a < x < b \tag{39}$$

The problem is to find a function $\varphi(x)$ on (a,b) that is bounded by $\beta(x)$ and $\gamma(x)$ to maximize an effectiveness functional $E(\varphi)$ given by the integral of a point-effectiveness function e. The functional $E(\varphi)$ is subject to a cost constraint $C(\varphi)$ which is the integral of a point-effectiveness function e. Wagner [15] gives theorems for both the differentiable and discrete cases of this problem. His theorem for the differentiable case is repeated.

Let e and c be real functions of two variables defined on

$$\left\{ (x, y) \mid a < x < b, \ \beta (x) \leqq \varphi (x) \leqq \gamma (x) \text{ and } -\infty < \varphi (x) < \infty \right. \qquad (40)$$

let their partial derivatives with respect to φ exist on (40), and denote these by

$$D_1 e$$

$$\text{and} \quad D_1 c \ .$$

For each fixed $x \in (a,b)$, assume that $D_1 e(\cdot,x)$ and $D_1 c(\cdot,x)$ are Riemann integrable on each bounded subinterval of $(\beta(x), \gamma(x))$. Let Φ be the set of all real-valued functions φ on (a,b) such that $\beta(x) \leq \varphi(x) \leq \gamma(x)$ for $a < x < b$ for which

$$-\infty < E(\varphi) \equiv \int_a^b e\left(\varphi(x), x\right) dx \quad \text{and}$$

$$-\infty < C(\varphi) \equiv \int_a^b c\left(\varphi(x), x\right) dx < \infty \qquad (41)$$

Suppose $g(x) \in \Phi$ has the property:

there exists a $\lambda > 0$ such that for all $x \in (a, b)$.

$$D_1 e(\cdot, x) \leqq \lambda D_1 c(\cdot, x) \quad \text{for} \quad g(x) < \varphi(x) < \gamma(x) \quad \text{and} \qquad (42)$$

$$D_1 e(\cdot, x) \geqq \lambda D_1 c(\cdot, x) \quad \text{for} \quad \beta(x) < \varphi(x) < g(x) \ . \qquad (43)$$

Then

$$E(g) = \max \left\{ E(\varphi) \mid \varphi \in \Phi \text{ and } C(\varphi) \leqq C(g) \right\} \qquad (44)$$

$$C(g) = \min \left\{ C(\varphi) \mid \varphi \in \Phi \text{ and } E(g) \geqq E(\varphi) \right\} \qquad (45)$$

This theorem states that for any chosen value of $\lambda > 0$, a function $g(x)$ which is in the feasible set of all functions Φ and which satisfies (42) and (43) will maximize the effectiveness integral, $E(\varphi)$, and minimize the cost $C(\varphi)$ for that particular value of $\lambda > 0$, hence it is cost effective. Wagner also points out that this theorem holds if there are multiple constraint, or cost, functions.

Using this theorem, it is seen that for the case of m constraint functions

$$E(g) - \sum_{i=1}^{m} \lambda_i C_i(g) \geq E(\varphi) - \sum_{i=1}^{m} \lambda_i C_i(\varphi) \tag{46}$$

The Lagrangian function is written as

$$L(\varphi, \lambda) = E(\varphi) - \sum_{i=1}^{m} \lambda_i C_i(\varphi) \tag{47}$$

For a given value of $\lambda > 0$, a minimization problem can be written as

$$\text{I} \qquad \min_{\lambda \in E_m^+} G(\lambda) \equiv \int_a^b \left[e\left(g(x), x\right) - \sum_{i=1}^{m} \lambda_i C_i \left(g(x)_1 x\right) dx + \alpha\lambda \right. \tag{48}$$

which is the dual problem of the maximization problem

$$\text{II} \qquad \max E(\varphi) \equiv \int_a^b E\left(\varphi(x), x\right) dx \tag{49}$$

$$\text{subject to} \quad C_i(\varphi) = \int_a^b c\left(\varphi(x), x\right) dx \leq \alpha_i \tag{50}$$

for all i = 1, ---, m

$$\text{and} \qquad \beta(x) \leqq \varphi(x) \leqq \gamma(x) \tag{51}$$

Problem I is the Wolfe dual of problem II. Problem II has a functional form very close to the search problem. As such, these problems have several dual properties:

(1) If φ is any feasible solution to problem II, and $\lambda \in E^{+}_{m}$,

$$E(\varphi) \leq G(\lambda) \tag{52}$$

(2) If there exists a feasible solution $g(x)$ to problem II and $\lambda \in E^{+}_{m}$ such that

$$\sum_{i=1}^{m} \lambda_i \left[\int_a^b c_i \left(g(x), x \right) dx - \alpha_i \right] = 0 \tag{53}$$

$$D_1 e(\varphi, x) - \sum_{i=1}^{m} \lambda_i D_1 c_i (\varphi, x) \leq 0 \tag{54}$$

whenever

$$g(x) < \varphi(x) < \gamma(x) \tag{55}$$

$$D_1 e(\varphi, x) - \sum_{i=1}^{m} \lambda_i D_1 c_i (\varphi, x) \geq 0 \tag{56}$$

whenever

$$\beta(x) < \varphi(x) < g(x) \tag{57}$$

then λ^o is a solution to the primal problem, $g(x)$ is an optimal solution to problem II, and

$$E(g) = G(\lambda^o) \tag{58}$$

(3) If g(x) is a solution to the dual problem then there exists a point λ in E_m^+ for which

$$\sum_{i=1}^{m} \lambda_i \left[\int_a^b c_i \left(g(x), x \right) dx - \alpha_i \right] = 0 \tag{59}$$

$$D_1 e(\varphi, x) - \sum_{i=1}^{m} \lambda_i D_1 c_i (\varphi, x) \leq 0 \tag{60}$$

whenever

$$g(x) < \varphi(x) < \gamma(x) \tag{61}$$

$$D_1 e(\varphi, x) - \sum_{i=1}^{m} \lambda_i D_1 c_i (\varphi, x) \geq 0 \tag{62}$$

whenever

$$\beta(x) < \varphi(x) < g(x) \tag{63}$$

These properties have been proven in [14].

Rewriting these results in terms of the search problem is done by the following replacement

$$e\left(\varphi(x), x\right) = p(x)\left[1 - \exp\left(-\varphi(x)\right)\right] \tag{64}$$

$$c(\varphi(x), x) = \varphi(x) \tag{65}$$

$$0 \leqq x \leqq X < +\infty \tag{66}$$

$$0 \leqq \varphi(x) \leqq M \tag{67}$$

$$p(x) \geqq 0 \tag{68}$$

$$C\left(\varphi C(\varphi)\right) = \int_0^X \varphi(x)\, dx \leq \alpha \tag{69}$$

The minimization and maximization problems are

$$\text{I} \qquad \underset{\lambda \epsilon E^+_m}{\text{Min}} \int_0^X \left[p(x)\left[1 - \exp\left(-g(x)\right)\right]\right] dx - \lambda \left[\int_0^X g(x)\, dx - \alpha\right] \tag{70}$$

subject to

$$p(x)\exp\left(-g(x)\right) - \lambda \leqq 0 \quad \text{if } g(x) < \varphi(x) < M \tag{71}$$

$$p(x)\exp\left(-g(x)\right) - \lambda \geqq 0 \quad \text{if } 0 < \varphi(x) < g(x) \tag{72}$$

$$\text{II} \qquad \operatorname*{Max}_{\varphi\epsilon\Phi} P(\varphi) = \int_0^X p\,(x) \left[1 - \exp\left(-\varphi\,(x)\right)\right] dx \tag{73}$$

subject to

$$\int_0^X \varphi\,(x)\; dx \leqq \alpha \tag{74}$$

$$0 \leqq \varphi\,(x) \leqq M \tag{75}$$

where α is a constraint on the total amount of search effort Q available and M is a maximum amount allocated to a particular region between x and x+ dx. So far, the preventive patrol problem has only considered one constraint, so E_m is one dimensional. In this case, search is conducted along a straight line over a finite portion from the origin to a point $X < +\infty$

From problem I, we see that

$$\lambda \left[\int_0^X g\,(x)\; dx - \alpha \right] \tag{76}$$

must be minimized. For a value of $\lambda > 0$, the minimum is attained if

$$\int_0^X g\,(x)\; dx = \alpha \tag{77}$$

or if the allocation of search effort uses all of the available resource. The necessary conditions for the optimal allocation function in I can be written as

$$g\,(x) = \begin{cases} \ln p\,(x) - \ln \lambda & \text{if } \lambda < p\,(x) \leqq \lambda \exp\,(M) \\ 0 & \text{if } 0 \leqq p\,(x) \leqq \lambda \\ M & \text{if } p\,(x) \geqq \lambda \exp\,(M) \end{cases} \tag{78}$$

The remaining problem is to find the optimal value of λ which would permit us to solve for g(x) and have

Problem I = Problem II.

The optim... value of $\lambda > 0$ is found by using the solution for g(x) in the constraint equation

$$\int_{0}^{X} g(x)\, dx = \alpha \qquad (79)$$

and obtaining

$$\int_{E_M} M dx + \int_{E_o} \left[\ell n\, p(x) - \ell n\, \lambda \right] dx = \alpha \qquad (80)$$

or

$$\frac{1}{L} \left[-\alpha + M \int_{E_M} dx + \int_{E_o} \ell n\, p(x)\, dx \right] = \ell n\, \lambda \qquad (81)$$

where

$$E = \left\{ x \mid \varphi(x) > 0 \right\} \qquad (82)$$

$$E_M = \left\{ x \mid p(x) \geqq \lambda\, e^M \right\} \qquad (83)$$

$$L = \int_{E_o} dx \qquad (84)$$

$$E_o = E - E_M \qquad (85)$$

The optimal value of λ can be found from the trancendental equation, and this value used to find the values of g(x) that maximizes problem II.

If the different weighting factors discussed before are used in the preventive patrol problem, the maximization and minimization problems become

IW
$$\underset{\lambda \in E_M^+}{\text{Min}} \int_0^X \left[p(x)v(x) \left(1 - \exp(-q(x)g(x))\right)\right] dx - \lambda \left[\int_0^X w(x)g(x)dx - \alpha\right] \quad (86)$$

subject to

$$p(x)v(x)q(x)\exp\left(-q(x)g(x)\right) - \lambda \int_0^X w(x)\ dx \leqq 0 \quad (87)$$

$$\text{if} \quad g(x) < \varphi(x) < M \quad (88)$$

$$p(x)v(x)q(x)\exp\left(-q(x)g(x)\right) - \lambda \int_0^X w(x)\ dx \geqq 0 \quad (89)$$

$$\text{if} \quad 0 < \varphi(x) < g(x) \quad (90)$$

IIW
$$\underset{\varphi \in \Phi}{\text{Max}} \int_0^X p(x)v(x) \left[1 - \exp\left(-q(x)g(x)\right)\right] dx \quad (91)$$

subject to

$$\int w(x)\ \varphi(x)\ dx(x) \leqq \alpha \quad (92)$$

$$\text{for } p(x) \geqq 0,\ v(x) \geqq 0,\ q(x) \geqq 0,\ w(x) \geqq 0 \quad (93)$$

$$0 \leqq x \leqq X < +\infty \quad (94)$$

$$0 \leqq w(x)\ \varphi(x) \leqq M \quad (95)$$

The value to the decision maker of this formulation comes from the economic interpretation of the primal problem variable λ. For this particular problem type (nonlinear Neyman-Pearson problem), the solution g(x) is the most cost-effective or efficient for a given value of λ. By this we mean that for a given value of $\lambda > 0$, g(x) is the allocation of effort that maximizes the probability of space-time coincidence for the least cost, but we also know that the total amount of resources (or cost) α are expended for the optimal solution. The value of λ is the marginal increase in the total probability of space-time coincidence for an added unit of the resource α. So, the value of λ tells the decision maker how much additional search ability he will obtain if he expends an additional unit of his allocation effort.

A final model will be discussed now that has been worked on by Stone [17]. This is the uncertain sweep rate problem. It refers to the case where some, or all of the parameters such as S, ρ, or T are random variables with a known density function. This formulation is useful if we use an allocation method based on past history. For instance, historical data that provided a measure of the probability of a police patrol unit detecting a criminal event could be used instead of a model explicitly considering the speed of patrol, the number of miles of streets and alleys, and the average availability of patrol units. The historical data would provide an empirical distribution giving the probability of detection as a function of patrol manpower. This data would probably be grouped by time of day, types of crimes, and area of the city.

For those cases where p(x), w(x), v(x), and q(x) are greater than zero,

$$g(x) = \frac{1}{q(x)} \left[\ln \left(p(x)q(x)v(x) \right) - \ln \int_o^X w(x)dx - \ln\lambda \right]$$

$$\text{if } \lambda' < p(x)v(x)q(x) < \lambda' \exp \left[\frac{Mq(x)}{w(x)} \right]$$

$$0 \text{ if } 0 \leqq p\ (x)v(x)q(x) \leqq \lambda' \qquad (96)$$

$$\frac{M}{w(x)} \text{ if } p(x)v(x)q(x) \geqq \lambda' \exp \left[\frac{Mq(x)}{w(x)} \right]$$

where $\lambda' = \lambda \int_o^X w\ (x)\ dx$ (97)

and the trancendental equation used to solve for the optimal value of λ' is

$$M \int_{E_M} \frac{dx}{w(x)} + \int_{E_o} \left[\frac{\ln \left(p(x)q(x)v(x) \right) - \ln \lambda'}{q(x)} \right] dx = \alpha \qquad (98)$$

where

$$E_M = \left\{ x \mid p(x)\ v\ (x)\ q\ (x) \geq \lambda' \exp \left[\frac{Mq(x)}{w(x)} \right] \right. \qquad (99)$$

$$E = \left\{ x \mid w(x)\ g\ (x) > 0 \right\} \qquad (100)$$

$$E_o = E - E_M \qquad (101)$$

Consider preventive patrol and write the problem as

$$\text{Max } P(\varphi) = \int_0^X p(x)\ b\left(\varphi(x),\ w\right) dx\ H\ (dw) \qquad (102)$$

where H is the prior distribution of $\underset{\sim}{w}$, the patrol unit sweep width. The function b is a local effectiveness function. If we have $\varphi(x) = k$, then $b(k,w)$ is the probability of finding the target given that it is located at x and $w = \underset{\sim}{w}$. Here, we may think of k as the number of preventive patrol units. We get the expression

$$\text{Max } P\ (\varphi) = \int_0^X p(x)\ B\ (k)\ dx \qquad (103)$$

where

$$B(k) = \int b\ (k,w)\ H\ (dw) \qquad (104)$$

$$\text{for } k \geq 0$$

subject to

$$\int_0^X \varphi\ (x)\ dx \leq \alpha \qquad (105)$$

This is the same form (a Neyman-Pearson problem) as before and the conditions for solution are the same. Namely,

$$p(x)\ B'\ (k) \geq \lambda \qquad \text{for } k < g\ (x) \qquad (106)$$

$$p(x)\ B'\ (k) \leq \lambda \qquad \text{for } g\ (x) < k \qquad (107)$$

$$\int_0^X g(x)\ dx = \alpha \qquad (108)$$

We assume that B(k) is concave, strictly increasing and B(0) = 0. Then $B'(k)$ is continuous, positive and strictly decreasing. Under these conditions there exists an inverse function $B'^{-1}(k)$ that is continuous and strictly decreasing. Stone [16] proves that there exists a λ such that for $p(x) > 0$

$$g(x) = \begin{cases} B'^{-1}\left(\frac{\lambda}{p(x)}\right) & \text{if } B'(0)\ p(x) \geq \lambda \\ 0 & \text{if } B'(0)\ p(x) < \lambda \end{cases} \qquad (109)$$

and $\int g(x)\ dx = \alpha$

This can be proven for any a priori probability density p(x) that is a non-negative integrable function on X -- conditions met by any probability density function defined over the real line. B can be any bounded function defined on E^+ such that B' is positive, continuous and strictly decreasing.

In the last developments for the primal and dual problems, the continuous form of the equations were used. To assist the reader in solving these problems with the use of a digital computer (or even a desk calculation if the problem is small) a discrete form of the equations will be used. The method of solution is a more general version of the method previously described and originally done by Charnes and Cooper.

In the discrete form, the necessary conditions for the optimal allocation function of problem I are

$$g_j = \begin{cases} \ln p_j - \ln \lambda & \text{if } \lambda < p_j \leqq \lambda e^M \\ 0 & \text{if } 0 \leqq p_j \leqq \lambda \\ M & \text{if } p_j \geqq \lambda e^M \end{cases} \qquad (110)$$

where M is an arbitrary limit (less than the overall amount of search resource available) on the amount of search effort expended on any region x_j. If this limit were not set, the conditions would be

$$g_j = \begin{cases} \ln p_j - \ln \lambda & \text{if } p_j > \lambda \\ 0 & \text{if } p_j \leqq \lambda \end{cases} \qquad (111)$$

Three cases can occur: (1) an amount of search $M\sum_{r \epsilon E_m} x_r = \alpha$ is allocated to r sections, (2) an amount of search $M\sum_{r \epsilon E_m} x_r + \sum(\ln p_s - \ln \lambda)x_s = \alpha$ is allocated to r+s sections, (3) and an amount of search $\sum_{s \epsilon E_o}(\ln p_s - \ln \lambda)x_s = \alpha$ is allocated to s sections. The solution method will consider all three cases. From the necessary conditions for an optimal allocation we will allocate search effort only to those regions where

$$\ln p_j > \ln \lambda \tag{117}$$

or

$$\ln p_j > \frac{1}{\sum_{s \epsilon E_o} x_s} \left[\sum_{s \epsilon E_o} \ln p_s \, x_s - \alpha + M\sum_{r \epsilon E_m} x_r \right. \tag{118}$$

Unless case (1) occurs, the optimal amount of search allocated to a sector is

$$g_j = \ln p_j - \frac{1}{\sum_{s \epsilon E_o} x_s} \left[\sum_{s \epsilon E_o} \ln p_s \, x_s - \alpha + M\sum_{r \epsilon E_m} x_r \right] \tag{119}$$

for Case (2)

and

$$g_j = \ln p_j - \frac{1}{\sum_{S \epsilon E_o} x_s} \left[\sum_{s \epsilon E_o} \ln p_s \, x_s - \alpha \right] \tag{120}$$

for Case (3).

The constraint equation is

$$\sum_{j \in E} g_j \quad x_j = \alpha \tag{112}$$

where $$E = \left\{ x_j \mid g_j > 0 \right\} \tag{113}$$

so the trancendental equation obtained by substituting the values of g_j in the constraint is

$$M \sum_{r \in E_M} x_r + \sum_{s \in E_o} (\ell n \; p_s - \ell n \; \lambda) \; x_s = \alpha \tag{114}$$

or

$$\frac{1}{\sum_{s \in E_o} x_s} \left[M \sum_{r \in E_M} x_r + \sum_{s \in E_o} x_s \; \ell n \; ps \quad - \alpha \right] = \ell n \; \lambda \tag{115}$$

where $$E_M = \left\{ X_r \mid p_r \geqq \lambda e^M \right\} \text{ and } E_o = E - E_M \tag{116}$$

Before proceeding, we will show that if

$$\ell n\ p_{s+1} > \frac{1}{\sum\limits_{S\epsilon E_o} x_s} \left(\sum_{s\epsilon E_o} \ell n\ p_s\ x_s - \alpha + M \sum_{r\epsilon E_m} x_r \right) \tag{121}$$

then

$$\ell n\ p_{s+1} > \frac{1}{\sum\limits_{s+1\epsilon E_o} x_{s+1}} \left(\sum_{\substack{s\epsilon E_o \\ s+1}} \ell n\ p_{s+1}\ x_{s+1} - \alpha + M \sum_{r\epsilon E_m} x_r \right) \tag{122}$$

since

$$\ell n\ p_{s+1} \left(\sum_{s\epsilon E_o} x_s \right) + \ell n\ p_{s+1}\ x_{s+1} > \sum_{s\epsilon E_o} \ell n\ p_s\ x_s + \ell n p_{s+1}\ x_{s+1} - \alpha + M \sum_{r\epsilon E_m} x_r \tag{123}$$

by the original hypothesis so

$$\ell n\ p_{s+1} > \frac{1}{\sum\limits_{s+1\epsilon E_o} x_{s+1}} \left[\sum_{s+1\epsilon E_o} \ell n\ p_{s+1}\ x_{s+1} - \alpha + M \sum_{r\epsilon E_m} x_r \right. \tag{124}$$

which obviously holds also in the case of $E_m=0$.

If this condition is true, region s + 1 is accepted for search, otherwise search is conducted only in the first s = r + 1 regions.

Allocation to regions greater than is made according to equation (110).

The algorithm for solving the problem with weighting factors q_i, w_i would be developed in a similar manner. For instance, the discrete form of the trancendental equation would be

$$\frac{1}{\sum_{s \epsilon E_o} x_s} \left[M \sum_{r \epsilon E_M} \frac{x_r}{w_r} + \sum_{s \epsilon E_o} \frac{x_s}{q_s} \ell n \left(p_s \; v_s \; q_s \right) - \alpha \right] = \ell n \; \lambda' \tag{125}$$

where

$$E_M = \left\{ x_r \mid p_r \; v_r \; q_r \geq \lambda' \; e^{\frac{Mq_r}{w_r}} \right\} \tag{126}$$

$$E = \left\{ x \mid w_j \; g_j > 0 \right\} \tag{127}$$

$$E_o = E - E_M \tag{128}$$

In this case the natural logarithms of the products p_j v_j q_j would be ranked in decreasing order. Similar minor changes to the algorithm would give the solution to the case where weightings on each region for search were given.

As a first step, the a priori probabilities p_j of a criminal event occurring in the j-th section are ranked in decreasing order, with p_1 representing the highest value, and p_2 the next highest. Next, compute the natural logarithms of the p_j. The section corresponding to p_1 will always receive some search effort if $\alpha > 0$. To determine the remaining sections for search - if any - the following steps are used.

1. Determine if $\ln p_1 > Mx_1 - \alpha$
2. If $\ln p_1 \leqq Mx_1 - \alpha$, then $E_M = 0$
3. In case (2.) occurs, then determine if $\ln p_2 > \frac{1}{X_1} \ln p_1 x_1 - \alpha$ and add region 2 to the search area if this is true, since if true then

   $$\ln p_z > \frac{1}{\sum_{s=1}^{2} x_s} \left(\sum_{1}^{2} \ln p_s x_s - \alpha \right)$$

4. Continue until a region j is found where

   $\ln p_j \leqq \frac{1}{\sum_{1}^{j-1} x_s} \left(\sum_{1}^{j-1} \ln p_s x_s - \alpha \right)$. At this point we know that only the first j-1 regions will receive any search effort. The actual amount will be determined by equation (110).

5. In case $\ln p_1 > Mx_1 - \alpha$, then the first search region receives an amount M. Similarly, if $\ln p_2 > M (x_1 + x_2)$ the next region also receives M search effort. The process continues until a region r + 1 is found such that $\ln p_{r+1} \leqq M \left(\sum_{1}^{r} x_r \right) - \alpha$.
6. Unless $M \left(\sum_{1}^{r} x_r \right) - \alpha = 0$, we know that some effort less than M will be allocated to the region r + 1.
7. Set s = r + 1, then determine if $\ln p_{s'+1} > \frac{1}{x_s} \ln p_s x_s + M \sum_{1}^{r} x_r - \alpha$.

APPENDIX B

SOME PRELIMINARIES

Let A_o, A_1, ··· denote the decisions made at times 0, 1, 2, ··· during the patrol. That is, A_τ denotes the decision as to what location in the sector to patrol at time $\tau + 1$. For example, using Figure 1, if the patrol car at time τ is in location 17, $X_\tau = 17$, then A_τ is either 5 or 13 or 17. We let K_i denote the set of possible decisions when the patrol car is in location i. For example, in Figure 1, if $X_\tau = 7$, then $K_{X_\tau} = K_7 = \{9\}$ ·, if $X_\tau = 16$, then $K_{X_\tau} = K_{16} = \{2, 11\}$ and so forth. Next, let $H_\tau = (X_o, A_o, X_1, A_1, \cdots, X_\tau, A_\tau)$ denote the sequence of locations patrolled and decisions made up to and including time τ (=0, 1, 2, ···).

Let

$$D_a (H_{\tau-1}, X_\tau) = Pr \{A_\tau = a \mid H_{\tau-1}, X_\tau\}$$

= probability (conditional) that the <u>decision</u> made is to patrol location a in K_{X_τ} at time $\tau + 1$, given the past sequence H_τ of locations patrolled and decisions made up to time τ, and given the present location X_τ of the patrol car at time τ.

Since these are probabilities, they must satisfy the conditions

$$D_a (H_{\tau-1}, X_\tau) \geqq 0 \qquad (3)$$

and

$$\sum_{a \in K_{X_\tau}} D_a (H_{\tau-1}, X_\tau) = 1$$

for all time τ = 0, 1, ..., and all possible sequences of past "histories" $H_{\tau-1}$ and locations in the sector, X_τ.

We now define what is meant by a patrol policy. A patrol policy is a set of random variables P of the form $D_a\ (H_{\tau-1}, X_\tau)$ satisfying conditions (3). That is, P is a patrol policy if

$$P = \left\{ D_a\ (H_{\tau-1}, X_\tau) : a \in K_{X_\tau},\ \tau = 0, 1, 2, \ldots \right\}$$

where the random variables in P satisfy the conditions in (3). A patrol policy is a procedure for making decisions at each point in time (deciding what location to patrol next). A patrol policy allows decisions to be made by the use of a "random" mechanism. That is, a patrol policy specifies for any time τ and any location X_τ of the patrol car in the sector, a probability distribution on the set of locations to patrol in time period $\tau + 1$.

Note that a patrol policy as defined above allows decisions to be made which are functions of the entire past history of the patrol. We will restrict our attention to patrol policies which are independent of past history (random or Markovian policies). That is, a patrol policy P is said to be random or Markovian if the functions $D_a\ (H_{\tau-1}, X_\tau)$ in P satisfy the conditions

$$D_a\ (H_{\tau-1}, X_\tau) = \Pr \left\{ A_\tau = a \mid H_{\tau-1}, X_\tau \right\}$$

$$= \Pr \left\{ A_\tau = a \mid X_\tau \right\}$$

for all $a \in K_{X_\tau}$ and $\tau = 0,\ \ , \ldots$.

In fact, it can be shown that Markovian policies are indeed the "best" or "optimal" for the models to be presented in the next section.

CONTINUED

1 OF 2

We will use the following notation to denote Markovian patrol policies, we say a patrol policy P^M is Markovian where

$$P^M = \left\{ D_{ia}: \ a \in K_i, \ i = 0, 1, \ \cdots, M \right\}$$

if

$$D_{ia} = \Pr\left\{ A_\tau = a \mid H_{\tau-1}, X_\tau = i \right\}$$

for all times $\tau = 0, 1, \cdots$ and all $a \in K_i$, for $i = 0, \cdots, M$, where the D_{ia}'s satisfy the conditions (analogous to (3))

$$D_{ia} \geqq 0, \text{ for all } a \in K_i \text{ and } i = 0, 1, \cdots, M,$$

and (4)

$$\sum_{a \in K_i} D_{ia} = 1 \text{ for all } i = 0, 1, \cdots, M.$$

We next define the laws of movement of the patrol car in the sector. These laws coupled with a Markovian patrol policy of the form P^M above will then give us a patrol matrix (transition) as discussed in the preceding section. Our objective is to construct by the use of the models presented in the next section an optimal patrol policy P^M which then determines, using the laws of movement in the sector as defined below, an optimal patrol matrix P^o. For this purpose, assume the following <u>random laws of movement</u> for the patrol car in the sector.

First, let

$$q_{ij}(a) = \Pr\left\{ X_{\tau+1} = j \mid X_\tau = i, A_\tau = a \right\}$$

for $a \in K_i$ and $i, j = 0, 1, \cdots, M$.

For any time $\tau (= 0, 1, \cdots)$, $q_{ij}(a)$ is the conditional probability that location j is patrolled in time period $\tau + 1$, given that at time τ, the patrol car is in location i and the decision made is to next patrol location a.

We require that the q_{ij} (a)'s satisfy the following conditions (5) - (9), due to the physical nature of the sector, the restrictions that U-turns are not allowed, and the properties of the patrol decision process.

First, since movement from location to location in the sector must always occur, the relation

$$\sum_{j=0}^{M} q_{ij}(a) = 1 \tag{5}$$

must hold for any decision a in K_i and for any location i (= 0, 1, ..., M). (5) simply states that regardless of the decision made when the patrol car is in location i, movement to a new location always occurs with probability 1.

Next, if the patrol car is in location i (=0, 1, ..., M) at any time period, and the decision made is to patrol location a in the next time period, then we set

$$q_{ij}(a) = 1 - \epsilon \text{ for } a = j, \text{ provided } a \in K_i, \tag{6}$$

$$q_{ij}(a) = \epsilon/M \text{ for all } a \neq j, \text{ provided } a \in K_i \tag{7}$$

$$q_{ij}(a) = \epsilon \quad \text{for } a = j \text{ and } a \text{ not in } K_i, \tag{8}$$

$$q_{ij}(a) = \frac{1 - \epsilon}{M} \quad \text{for } a \neq j \text{ and } a \text{ not in } K_i, \tag{9}$$

where ϵ is an arbitrarily "small" positive number.

Condition (6) guarantees that movement is almost surely made from location i to location j provided the decision made, a, is equal to j and j is in K_i. If M is large and ϵ is chosen to be "very close" to zero but positive, then (7), (8), and (9) almost surely guarantee that infeasible movements between locations are never made. The use of the small positive number ϵ implies a certain probabilistic structure (recurrent Markov matrix) which insures that the models presented are solvable.

Next assume we are given a patrol policy not necessarily optimal, say

$$P^M = \left\{ D_{ia} : \ i, a = 0, 1, \ldots, M \right\}$$

and consider any two locations i and j in the sector. By the definition of the transition probabilities in a patrol matrix, we have for any τ (=0, 1, ...),

$$\begin{aligned} p_{ij} &= \Pr\left\{ X_{\tau+1} = j \mid X_\tau = i \right\} \\ &= \sum_a \Pr\left\{ X_{\tau+1} = j \mid X_\tau = i, A_\tau = a \right\} \times \Pr\left\{ A_\tau = a \mid X_\tau = i \right\} \quad (10) \\ &= \sum_{a=0}^{M} q_{ij}(a) \ D_{ia} \ . \end{aligned}$$

First consider the case where j is in K_i. That is, assume j is a feasible location to patrol upon leaving location i. We have

$$p_{ij} = D_{i(a=j)} - \frac{\epsilon\ (M+1)}{M} \ D_{i(a=j)} - \frac{(1-2\epsilon)}{M} \sum_{a \in K_i} D_{ia} + \frac{1 - \epsilon}{M} \quad (11)$$

It is easily shown that

$$D_{i(a=j)} - \epsilon \leqq p_{ij} \leqq D_{i(a=j)} + \frac{1}{M} \quad (12)$$

Hence for M large and ϵ small, p_{ij} is approximately equal to $D_{i(a=j)}$, the probability the next location patrolled is j (j is in K_i), given that we have finished patrolling location i.

For j not in K_i we have

$$p_{ij} = \left(\epsilon - \frac{1-\epsilon}{M}\right) D_{i\ (a=j)} - \frac{(1-2\epsilon)}{M} \sum_{a \in K_i} D_{ia} + \frac{1-\epsilon}{M}, \qquad \text{and} \tag{13}$$

$$\frac{1}{M}\left(\epsilon - (1-\epsilon)\, D_{ij}\right) + \epsilon D_{ij} \leqq p_{ij} \leqq \frac{(1-\epsilon)}{M}\left(1 + D_{i\ (a=j)}\right) + \epsilon D_{ij} \tag{14}$$

which for M relatively large and ϵ small, p_{ij} is close to zero for all i and j, j not in K_i.

Note that given any patrol policy of the form P^M (we will show how to construct optimal patrol policies in the next section), one can then construct an optimal patrol matrix P^o using formulas (11) and (13).

We now turn to a discussion of some other parameters and random variables which will be used, for the most part, in the optimization models presented in the next section.

First, for each street i (=1, ···, N) in the sector, let L_i be the minimal acceptable coverage rate. In many instances, one would expect L_i to be either zero or one for streets i having a low proportion of crime incidents relative to the remaining streets in the sector.

Next, let S denote the length of the shift in hours for the sector. One would expect that S would be equal to the normal shift time minus twice the time it takes the patrol car to traverse location 0. Letting s = average speed of patrol (in miles per hour) in the sector, and ℓ = average length of a street in the sector, we set

$$S^h = (\text{effective shift time}),$$

$$= \frac{s}{\ell} \times S .$$

s^h is an estimate of the largest number of streets that can be patrolled during a particular shift and is also based on the actual physical characteristics of location zero (0) in the sector.

Next, for any location in the sector, say ℓ, we let C_ℓ be the Bernoulli random variable denoting the occurrence or non-occurrence of a crime or incident, and set

$$\Pr\left\{ C_\ell = 1 \right\} = P_r\left\{ \text{incident occurs in } \ell \right\} = c_\ell$$

and

$$\Pr\left\{ C_\ell = 0 \right\} = P_r\left\{ \text{no incident occurs in } \ell \right\} = 1 - c_\ell .$$

We assume we have M such random variables with known distributions (C_ℓ and $1 - C_\ell$) for each location ℓ (=0, 1, ..., M) in the sector.

Note that one would expect that the pair of random variables C_i and C_{i+1} (i = 1, ..., N = number of streets in the sector) have the same distribution ($C_i = C_{i+1}$ i = 1, ..., M-1). In many instances it may be quite difficult to determine the distributions of the C_i's and C_{i+1}'s, since the data (available) may be in a form which does not distinguish which side of the street crimes (or incidents) occur. However, it may be appropriate and desirable for some streets in the sector, say for example street i, to estimate both C_i and C_{i+1}, because of the nature of the preventive patrol (one-man, two-man cars, etc.), the physical characteristics of street i, and the incident statistics for street i. Also, it should be noted that

$$\text{Expected value of } C_\ell = E\left[C_\ell\right]$$

$$= \Pr\left\{ C_\ell = 1 \right\} = c_\ell, \qquad (15)$$

for ℓ = 1, ..., M.

An obvious criticism of the random variables (this is true of most, if not all, of the preventive patrol models using random variables) is that they are time independent. However, because of the nature of the objective functions used in our models and the fact we will be looking at the patrol decision process as a renewal process, that is, the time between sector sweeps is a recurrent event, the time independent properties of the random variables C_ℓ ($\ell = 1, \ldots, M$) may be a good approximation. If not, the models can be modified to incorporate time dependent random variables. We will say more about this in the next section.

Next let T_o be the random variable which denotes the time the patrol car spends patrolling the sector prior to the first return to location 0. T_o-1 is then the number of locations patrolled between the times location 0 is patrolled. That is, $X_o = 0$ (at the beginning of patrol) and since patrol evolves according to a patrol matrix P of transition probabilities, T_o is the first time τ ($\geqq 1$) such that $X_\tau = 0$. Mathematically,

$$T_o = \text{Min} \left\{ \tau : \ X_\tau = 0, \ \tau \geqq 1 \right\} \qquad (16)$$

Note that the probability distribution of T_o is completely determined given a patrol policy P^M (which determines a patrol matrix). We will usually require that the expected value of T_o be equal or less than S^h, the effective shift time.

We now turn to the statement and analysis of the preventive patrol schedule models:

MODEL I - THE ADDITIVE SCHEDULING AND ALLOCATION MODEL

Let $W_o, W_1, \cdots, W_{T_o}$ be the "benefits" or "returns" from preventive patrol in the sector in time periods $0, 1, \cdots, T_o$ where $T_o = \text{Min}\left\{ \tau : X_\tau = 0, \ \tau \geqq 1 \right\}$. We assume $W_o = 0$, $W_{T_o} = 0$, and for $\tau = 1, \cdots, T_o - 1$,

$$W_\tau = \begin{cases} 1 \text{ if } X_\tau = j \text{ and } C_j = 1 \\ 0 \text{ if } X_\tau = j \text{ and } C_j = 0 \end{cases} \tag{17}$$

for all locations j ($=1, \cdots, M$) in the sector S where $C_o \equiv 0$.

We assume that the random variables X_τ and therefore T_o are independent of. $C_1, \ldots, C_M$. That is, the occurrence or non-occurrence of an incident is independent of the location of the patrol car in the sector. Hence, given any patrol policy,

$$\begin{aligned} E\,[W_\tau] &= \sum_{j=1}^{M} \Pr\left\{ X_\tau = j,\ C_j = 1 \right\} \\ &= \sum_{j=1}^{M} \Pr\left\{ X_\tau = j \right\} \times \Pr\left\{ C_j = 1 \right\} \\ &= \sum_{j=1}^{M} C_j \times \Pr\left\{ X_\tau = j \right\}. \end{aligned} \tag{18}$$

Note that the terms in the sum (18), $C_j \times Pr\left\{X_\tau = j\right\}$, represent the probability of a <u>space-time</u> <u>coincidence</u> in location j during time period τ. Therefore, $E[W_t]$ is the <u>total</u> probability of a space-time coincidence in period τ during preventive patrol. It is this quantity, $E[W_t]$, that we want to maximize the sum of from period 0 up to period T_o subject to certain constraints. Note that we are assuming that a necessary condition for an arrest during time period τ in location j is that $X_\tau = j$ and $C_j = 1$. See Appendix I of the paper as well as the technical report by Larson [9] for a more detailed discussion of the ramifications of space-time coincidences during random preventive patrol.

It should also be noted that the particular value of $E[W_\tau]$ in time τ is dependent on the particular patrol policy being used. That is, $E[W_\tau]$ and therefore the probabilities $Pr\left\{X_\tau = j\right\}$ for any time period τ and location j is a function of the particular patrol policy being used. For example, let $P^M = \left\{D_{ia}: \; i, a = 0, \ldots, M\right\}$ be any patrol policy. At time $\tau - 1$ the patrol car is in some location $i_{\tau-1}$, $X_{\tau-1} = i_{\tau-1}$, and a decision is made to patrol location a_{t-1} in time τ, $A_{\tau-1} = a_{\tau-1}$, with probability $D_{i_{\tau-1}, a_{t-1}}$.

We then have for any location j in the sector,

$$\Pr\{X_\tau = j\} = \sum_{i_{\tau-1}} \sum_{a_{\tau-1}} \Pr\{X_\tau = j,\ X_{\tau-1} = i_{\tau-1},\ A_{\tau-1} = a_{\tau-1}\}$$

$$= \sum_{i_{\tau-1}} \sum_{a_{\tau-1}} \Pr\{X_\tau = j \mid X_{\tau-1} = i_{\tau-1},\ A_{\tau-1} = a_{\tau-1}\} \times$$

$$\Pr\{X_{\tau-1} = i_{\tau-1},\ A_{\tau-1} = a_{\tau-1}\}$$

$$= \sum_{i_{\tau-1}} \sum_{a_{\tau-1}} q_{i_{\tau-1},j}\left(a_{\tau-1}\right) \times D_{i_{\tau-1},a_{\tau-1}} \times \Pr\{X_{\tau-1} = i_{\tau-1}\} .$$

Continuing inductively in this manner gives

$$\Pr\{X_\tau = j\} = \sum_{i_{\tau-1}} \sum_{i_{\tau-2}} \cdots \sum_{i_o} \sum_{a_{\tau-1}} \cdots \sum_{a_o} q_{i_{\tau-1},j}\,(a_{\tau-1})\, D_{i_{\tau-1},\,a_{\tau-1}} \times$$

$$q_{i_{\tau-2},i_{\tau-1}}\,(a_{\tau-2})\, D_{i_{\tau-2},\,a_{\tau-2}} \cdots \times q_{i_o i_1}\,(a_o)\, D_{i_o a_o} \times \Pr\{X_o = 0\} .$$

The entire movement of the patrol car up to time τ and, hence $\Pr\{X_\tau = j\}$ and $E[W_\tau]$ are dependent on the particular patrol policy being used. We will now adopt the notation $P_{P^M}\{\cdot\}$ and $E_{P^M}[\cdot]$ to denote probability measures and expectations which are functions of random patrol policies P^M.

Next, let $N_j(T_o)$ be the number of times location j is patrolled during the first T_o time units. To exhibit the form of $N_j\ (T_o)$ more explicitly, let $I_j\ (X_\tau)$ (for $j = 0, 1, \cdots, M$ and $\tau = 0, 1, \cdots$) be the indicator random variables. Then

$I_j\ (X_\tau) = 1$ if $X_\tau = j$ and $I_j\ (X_\tau) = 0$ if $X_{\tau \neq j}$. Then we can write the random variables $N_j\ (T_o)$ in terms of the random variables $I_j\ (X_\tau)$ as follows:

$$N_j\ (T_o) = I_j\ (X_o) + I_j\ (X_1) + \cdots + I_j\ (X_{T_o})$$

$$= \sum_{\tau=0}^{T_o} I_j\ (X_t), \quad \text{where} \tag{19}$$

again $\qquad T_o = \text{Min} \left\{ \tau: X_\tau = 0,\ \tau \geqq 1 \right\}$.

Note that the probability distribution of the random variables $N_j(T_o)$ and T_o as well as their expectations are again dependent on the patrol policy being used.

We now turn to the statement of Model I.

MODEL I: $\underset{P^M \text{ in } M}{\text{Maximize}} \quad E_{P^M} \left[\sum_{\tau=0}^{T_o} W_\tau \mid X_o = 0 \right]$ (20)

subject to the conditions that

$$E_{P^M} \left[N_{2i}\ (T_o) + N_{2i-1}\ (T_o) \mid X_o = 0 \right] = L_i \quad (i = 1, \cdots, N) \tag{21}$$

$$E_{P^M} \left[T_o \mid X_o = 0 \right] \leqq S^h \tag{22}$$

where M is the set of all random or Markovian patrol policies. That is, the elements in the set M are of the form

$$P^M = \{ D_{ia}: i, a = 0, 1, \cdots, M \}$$

where $\qquad D_{ia} = P_{P^M} \{ A_\tau = a \mid X_\tau = i \}$,

$D_{ia} \geqq 0 \qquad$ for $\tau = 0, \cdots$, all a in K_i,

$\sum_a D_{ia} = 1 \qquad$ and all locations i in the sector S

In (20), the objective function, we want to determine a policy which during a sector sweep of duration T_o, maximizes the expected number of space-time coincidences.

The constraints in (21) require that the average number or expected number of times street i is patrolled, $E_{P^M} [N_{2i-1} (T_o) + N_{2i} (T_o) \mid X_o = 0]$, during a sector sweep of duration T_o, be equal or greater than the given lower bound L_i (for streets $i = 1, \cdots, N$). Next, constraint (22) requires that the average sector sweep time be no larger than the effective shift time. All of the expectations appearing in (I) require that preventive patrol begins in location 0.

Note that once the patrol car returns to location 0 (a sector sweep is completed or an apprehension is made requiring a return), preventive patrol would begin again with the parameters appropriately modified (decreased). It may be appropriate to recompute a new schedule based on new incident (crime) statistics (c_ℓ) at this time of return.

Following the approach of Derman [1], Model I is mathematically equivalent to

MODEL I*: Maximize P^M in M

$$\frac{1}{\pi_o(P^M)} \sum_{i=0}^{M} \sum_{a=0}^{M} \pi_i (P^M) D_{ia}^{P^M} C_a \tag{23}$$

subject to the conditions

$$\frac{\pi_{2i-1}(P^M)}{\pi_o (P^M)} + \frac{\pi_{2i} (P^M)}{\pi_o (P^M)} \geqq L_i \ (i = 1, \cdots, N) \tag{24}$$

and

$$\frac{1}{\pi_o(P^M)} \leqq S^h \tag{25}$$

In I*, (23) is equivalent to (20), (24) is equivalent to (21), and (25) is equivalent to (22). The $D_{ia}^{P^M}$'s (i, a = 0,•••, M) are the "decision" probabilities associated with a given patrol policy P^M.

To give an interpretation of the numbers $\pi_i(P^M)$, first let P^M be any patrol policy in M and set $\pi(P^M) = \left(\pi_o\ (P^M),\ \pi_1\ (P^M),\ \cdots,\ \pi_M\ (P^M)\right)$, (M+1) vector. Then, given a patrol policy P^M, and the resulting patrol matrix, say P^{P^M} constructed using P^M, (10), and (11), it can be shown that the $\pi_i(P^M)$'s satisfy uniquely the following system of linear equations.

$$\begin{aligned} \pi(P^M)\ P^{P^M} &= \pi(P^M) && \text{(M + 1 equations)} \\ \pi_o\ (P^M) + \pi_1(P^M) + \cdots + \pi_M\ (P^M) &= 1 && \text{(1 equation)} \end{aligned} \qquad (26)$$

For a given patrol policy P^M, $\pi_i\ (P^M)$ is the long-run average or steady state probability that the patrol car visits location i. In (24), $\frac{\pi_j(P^M)}{\pi_o(P^M)}$ is the expected number of visits to location j between visits to location 0.

The form of the objective function follows directly by using standard renewal theory arguments. For example, see Chung [18], Section 1.8, to show that

$$E_{P^M}\left[\sum_{\tau=0}^{T_o} W_\tau \mid X_o = 0\right] = \frac{1}{\pi_o(P^M)} \sum_{i=0}^{M} \sum_{a=0}^{M} \pi_i (P^M)\, D_{ia}^{P^M} c_a$$

where $\pi_o(P^M)$, $\pi_1(P^M), \cdots, \pi_M(P^M)$ satisfy the system of linear equations (26). Similarly, it can be shown (see Karlin [19], Chapter 5),

$$E_{P^M}[T_o \mid X_o = 0] = \frac{1}{\pi_o(P^M)} \qquad \text{and}$$

$$E_{P^M}[N_j (T_o) \mid X_o = 0] = \frac{\pi_j (P^M)}{\pi_o (P^M)}$$

for all locations $j = 0, 1, \cdots, M$.

One might attempt to solve Model I by enumerating all patrol policies which satisfy the constraints (24), (25), and (26) and then choose that policy which yields the highest value of the objective function, (23). However, this is clearly impossible, since the number of possible patrol policies is uncountable. The next model which is equivalent to I*, and hence to I does provide an efficient technique for determining the optimal patrol policy, and hence the optimal patrol matrix.

For this purpose let

$$x_{ia} = \pi_i \, (P^M) \, D_{ia}^{P^M} \qquad a, i = 0, 1, \cdots, M. \quad (27)$$

x_{ia} can be interpreted as the joint probability that the patrol car is in location i and the decision is to patrol location a in the next time period. Using the above transformation, we can now formulate Model I* as

MODEL I**: Find $\{x_{ia}: i, a = 0, 1, \cdots, M\}$

to

$$\text{Maximize} \quad \frac{\sum_{i=0}^{M} \sum_{a=0}^{M} x_{ia} \, c_a}{\sum_{a=0}^{M} x_{oa}} \qquad (28)$$

subject to the Constraints

$$\sum_{a=0}^{M} x_{2i-1,\, a} + \sum_{a=0}^{M} x_{2i,\, a} \geqq L_i \sum_{a=0}^{M} x_{0a} \qquad (29)$$

$$(i = 1, \cdots, N) \quad ,$$

$$S^h_{\cdot} \sum_{a=0}^{M} x_{oa} \geqq 1, \qquad (30)$$

$$\sum_{k=0}^{M} \sum_{a=0}^{M} q_{ki}(a) \; x_{ka} = \sum_{a=0}^{M} x_{ia} \tag{31}$$

$$(i = 0, 1, \cdots, M),$$

$$\sum_{i=0}^{M} \sum_{a=0}^{M} x_{ia} = 1, \tag{32}$$

$$x_{ia} \geqq 0 \qquad i, a = 0, 1, \cdots, M.$$

(28), (29), (30), (31), and (32) are obtained from (23), (24), (25), and (26) by making the transformation (27) and using the fact that for any patrol policy P^M,

$$\pi_i (P^M) = \sum_{a=0}^{M} x_{ia} \qquad (i = 0, \cdots, M), \text{ and}$$

the transition probabilities $P_{ij}^{P^M}$ in the patrol matrix P^{P^M} (associated with the policy P^M)

satisfy
$$P_{ij}^{P^M} = \sum_{a=0}^{M} q_{ij}(a) \; D_{ia}^{P^M}$$

$$\text{for } i, j = 0, \cdots, M.$$

Model I ** is a linear fractional programming problem with 3N + 1 constraints (N is the number of streets in the sector, M = 2N) and $4N^2$ + 4N + 1 variables. Fortunately the nonlinearity of (28) causes no computational problems, since it can be made linear by a transformation of variables. Hence linear programming computer codes can be used to solve I** and hence solve I, our original model.

Assume we have solved I** obtaining an optimal solution, say $\{x^o_{ia}, i, a = 0, \cdots, M\}$. Then the optimal patrol policy $\{D^o_{ia}, i, a = 0, \cdots, M\}$ is obtained by (27) and we have

$$D_{ia} = \frac{x^o_{ia}}{\sum_{j=0}^{M} x_{ij}} \qquad \text{for } i, a = 0, \cdots, M. \tag{33}$$

The optimal patrol matrix $P^o = (P^o_{ij})$ is then generated by the equations

$$P_{ij} = \sum_{a=0}^{M} q_{ij}(a) \; D^o_{ia} \tag{34}$$

Finally, it is easily shown [20] that I** can be transformed to a linear programming problem by setting

$$Y_{ia} = \frac{x_{ia}}{\sum_{a=0}^{M} x_{oa}} \qquad , i, a = 0, 1, \cdots, M,$$

and

$$Y_{N+1} = \frac{1}{\sum_{a=0}^{M} x_{oa}}$$

then we can write I** as a linear programming problem in $\{Y_{N+1}, Y_{ia}, i, a = 0, 1, \cdots, M\}$ as follows

$$\text{Maximize} \sum_{i=0}^{M} \sum_{a=0}^{M} Y_{ia} \, c_a$$

Subject to $\sum_{a=0}^{M} Y_{2i-1, a} + \sum_{a=0}^{M} Y_{2i, a} \geqq L_i \quad (i = 1, \cdots, N)$

$$Y_{N+1} \geqq 1$$

$$\sum_{k=0}^{M} \sum_{a=0}^{M} q_{ki}(a) \; Y_{ka} = 1$$

$$(i = 0, 1, \cdots, M)$$

$$\sum_{i=0}^{M} \sum_{a=0}^{M} Y_{ia} = Y_{N+1},$$

$$\sum_{a=0}^{M} Y_{oa} = 1.$$

$$Y_{ia} \geqq 0 \qquad i, a = 0, 1, \cdots, M.$$

Note that the above linear programming model has only one more constraint and one more variable then the fractional programming model I**. If, for example, a sector has 50 streets (N = 50), we would have to solve a linear programming problem with 152 constraints, which is not unreasonable considering existing linear programming computer codes.

Model II - THE GEOMETRIC SCHEDULING AND ALLOCATION MODEL

Let $W_1, \cdots, W_{T_o}$ be the "returns" or "benefits" from preventive patrol in the sector in time periods $1, \cdots, T_o$ where $T_o = \text{Min}\ \{\tau: X_\tau = 0, \tau \geqq 1\}$, $(X_o = 0)$. Furthermore, (as in Model I) let $N_j\ (T_o)$ be the number of times location j (= 1, $\cdots$, M) is patrolled between the times the patrol car is in location 0. We assume the random variables W_τ have the following form.

$$W_t = \begin{cases} 1 - e^{-\varphi_j(n)} & \text{if } X_\tau = j,\ C_j = 1,\ N_j\ (T_o) = n, \\ & \quad \text{and } \tau < T_o, \\ 0 & \text{if } X_\tau = j,\ C_j = 0,\ N_j\ (T_o) \text{ arbitrary}, \\ & \quad \text{and } \tau < T_o \end{cases} \qquad (35)$$

for all locations j in the sector S, where $\varphi_j(n)$ is a strictly increasing function. We set $C_o \equiv 0$ and assume the random variables X_τ $(\tau = 0, 1, \cdots)$ and $C_1, \cdots, C_M$ are independent with the probability distributions of the C_i's given by

$$\begin{aligned} \Pr\{C_j = 1\} &= \Pr\ \{\text{incident occurs in location } j\} \\ &= c_j\ (\geqq 0), \end{aligned}$$

and

$$\begin{aligned} P_r\{C_j = 0\} &= P_r\ \{\text{no incident occurs in location } j\} \\ &= 1 - c_j \end{aligned}$$

for all locations j (=1, $\cdots$, M).

Note that we are not assuming (as in Model I) that if the patrol car is in location j at time τ, $X_\tau = j$ and an incident is in progress, $C_j = 1$, a detection occurs, $W_\tau = 1$.

Rather, if $X_\tau = j$, and $C_j = 1$, $W_\tau = 1 - e^{-\varphi_j(n)} < 1$, given that the number of visits to state j during a sector sweep is n. Note that as n increases, W_τ is "close" to one.

We assume the following form for the functions $\varphi_j\,(\cdot)$ $(j = 1, \cdots, M)$.

$$\varphi_j\,(n) = \varphi_j \cdot n \qquad \left(N_j(T_o) = n\right) \qquad (36)$$

where

$$\varphi_j = \frac{s \cdot \tau\ j}{\ell \cdot E[T_o]} \quad ,$$

s = average speed of patrol in the sector, (miles per hour),

ℓ = average length of a street in the sector (in miles),

and τ_j = average crime duration in location j, $(j = 1, \cdots, M)$.

We will now discuss, briefly, the functions $\varphi_j(n)$ in (36). For a more detailed discussion, see Larson [6] or part I of this paper.

Let T be the effective sector sweep time, time in hours between visits to location 0. We set

$$T = \frac{E[T_o]}{s/\ell} = \frac{\varphi \cdot E[T_o]}{s} \quad ,$$

where s/ℓ = average number of streets patrolled per hour. Then letting $N_j(T_o) \cdot \tau_j$ be the (maximum) total time crimes or incidents in progress in location j when the patrol car is in location j, $N_j\,(T_o)$ times, we let

$$1 - e^{-N_j(T_o) \cdot \frac{\tau_j}{T}} = 1 - {}^{-N_j\,(T_o)\ \frac{s \cdot \tau_j}{\ell \cdot E[T_\cdot]}} \qquad (37)$$

$$= 1 - {}^{\varphi_j \left(N_j\ (T_o) \right)}$$

be the (approximate) probability of the patrol being in location j during an incident of duration τ_j.

Note that the expectations and probabilities in (35) and (36) are dependent (as in Model I) on the particular patrol policy used. Hence our problem is to determine that patrol policy P^M in M to maximize

$$E_{P^M}\left[\sum_{\tau=0}^{T_o} W_\tau \mid X_o = 0\right]$$

subject to the constraints

$$E_{P^M}\left[N_{2i}(T_o) + N_{2i-1}(T_o) \mid X_o = 0\right] \geqq L_i \qquad (i = 1, \cdots, N)$$

$$E_{P^M}\left[T_o \mid X_o = 0\right] = S^h$$

where the above constraints have the same interpretation as in Model I.

In Model II we are asked to determine that random patrol policy which maximizes the total probability of space-time coincidence, or equivalently, the number of detections during a sector sweep. In choosing a patrol policy in M we are in effect determining the number of visits to each location in the sector, which by (37), determines the probability of the patrol car being in each location, that is, the policy used determines $N_j(T_o)$ for all locations j (=1, $\cdots$, M). In this light we now reformulate our problem.

Let $R^1, R^2, \cdots, R^M$ be the <u>total</u> return from preventive patrol during a sector sweep of duration T_o in locations 1, 2, $\cdots$, M where $T_o = \text{Min}\{\tau: X_\tau = 0, \tau \geqq 1\}$ and $X_o = 0$. Let $N_j(T_o)$ be the number of times location j is patrolled during a sector sweep. We assume the random variables R', $R^2, \cdots, R^M$ have the following form.

$$R^j = R^j(n) = \begin{cases} 1 - e^{-\varphi_j(n)} \text{ if } N_j(T_o) = n \text{ and } C_j = 1, \\ 0 \text{ if } N_j(T_o) = n \text{ and } C_j = 0, \end{cases} \qquad (38)$$

for all locations j = 1, $\cdots$, M, where the functions $\varphi_j(n)$ are given by (36).

Note that since $C_1, \cdots, C_M$ are independent of the random variables $X_o, X_1, \cdots,$

the $E[R^j \mid X_o = 0] = E\left[R^j \left(N_j\ (T_o)\right) \mid X_o = 0\right]$

$$= c_j - E\left[c_j \varphi^{-\varphi_j \left(N_j(T_o)\right)} \mid X_o = 0\right].$$

Letting $C = \sum_{j=1}^{M} c_m$ our problem

can now be stated as

MODEL II: Maximize $\left\{C - E_{P^M}\left[\sum_{j=0}^{M} c_j e^{-\varphi_j \left(N_j(T_o)\right)} \mid X_o = 0\right]\right\}$

$$= C - \underset{P^M \text{ in } M}{\text{Minimum}}\ E_{P^M}\left[\sum_{j=0}^{M} c_j e^{-\varphi_j \left(N_j(T_o)\right)} \mid X_o = 0\right] \qquad (39)$$

subject to the conditions

$$E_{P^M}\left[N_{2i-1}\ (T_o) + N_{2i}\ (T_o)\right] \geqq L_i \ (i = 1, \cdots, N) \qquad (40)$$

$$E_{P^M}\left[T_o \mid X_o = 0\right] = S^h \qquad (41)$$

where M is the set of all random or Markovian patrol policies.

By use of certain probabilistic arguments, the authors have shown that Model II is mathematically equivalent to

MODEL II*

$$C - \underset{P^M \text{ in } M}{\text{Minimize}} \sum_{j=1}^{M} c_j \times \frac{\sum_{i=1}^{M} \overline{\pi}_{ij}(P^M) \left\{\sum_{a=0}^{M} q_{oi}(a)\ D_{oa}^{P^M}\right\}}{\overline{\pi}_j^2(P^M)\ (1 - e^{-\overline{\varphi}_j}) + \overline{\pi}_j\ e^{-\overline{\varphi}_j}} \qquad (42)$$

subject to the conditions

$$\frac{\pi_{2i-1}(P^M)}{\pi_o(P^M)} + \frac{\pi_{2i}(P^M)}{\pi_o(P^M)} \geqq L_i\ (i = 1, \cdots, N) \qquad (43)$$

and

$$\frac{1}{\pi_o(P^M)} = S^h \qquad (44)$$

In II*, for a given random patrol policy, say P^M in M, $\pi_i(P^M)$ are the steady state probabilities for locations i (=0, 1, •••, M) in the sector which satisfy, as in Model I, the system of linear equations given in (26). $\bar{\varphi}_j$ for j = 1, •••, M have the form, using (36) and (44),

$$\bar{\varphi}_j = \frac{s \cdot \tau_j \cdot \pi_o(P^M)}{\ell}$$

$$= \frac{s \cdot \tau_j}{\ell \cdot s^h}$$

The $\bar{\pi}_j(P^M)$'s and $\bar{\pi}_{ij}(P^M)$ have the following interpretation. First, $\bar{\pi}_j(P^M) = \bar{\pi}_{jj}(P^M)$ for j = 1, •••, M, where $\bar{\pi}_{ij}(P^M)$ is the expected number of times location j(≠0) is patrolled using policy P^M, starting in location i, during a sector sweep. That is $\bar{\pi}_{ij}(P^M)$ is the expected number of times location j(≠0) is patrolled, starting in location 0, prior to the first time the patrol car returns to location 0. The $\bar{\pi}_{ij}(P^M)$'s are easily shown to satisfy the following system of linear equations, uniquely.

$$\bar{\pi}\,(P^M)\; Q^{P^M} = \dot{\bar{\pi}}(P^M) \tag{45}$$

where $\bar{\pi}(P^M) = \left(\bar{\pi}_{ij}(P^M)\right)$, the MxM matrix of expected values, and Q^{P^M} is the MxM matrix of transition probabilities obtained by deleting the first row and the first column of the patrol matrix P^{P^M}.

The seemingly complicated nature of the objective function (42) is due to the fact that in (39), we are computing the expected values of non-linear random functions. Unfortunately, since the functions $e^{-\varphi_j(n)}$ are convex, we cannot substitute expected values in place of $N_j(T_o)$, appearing in the arguments of $e^{-\varphi_j(n)}$, in order to attempt a good approximation.

Next, let

$$x_{ija} = \bar{\pi}_{ij}(P^M)\, D_{ja}^{P^M} \qquad (i, j = 1, \cdots, M) \quad (a = 0, 1, \cdots, M) \tag{46}$$

$$x_{ia} = \pi_i (P^M)\, D_{ia}^{P^M} \qquad (i, a = 0, \cdots, M). \tag{47}$$

Then II* is equivalent to

MODEL II**: Find $\{x_{ija}:\ i, j = 1, \cdots, M;\ a = 0, 1, \cdots, M\}$ and $\{x_{ia}:\ i, a = 0, 1, \cdots, M\}$ which are optimal for the problem

$$\text{C - Minimize} \sum_{j=1}^{M} c_j \left\{ \frac{S^h \sum_{a=0}^{M} x_{ja}}{\left(\sum_{a=0}^{M} x_{jja}\right)^2 \left(1-e^{-\bar{\varphi}_j}\right) + \sum_{a=0}^{M} x_{jja} \cdot e^{-\bar{\varphi}_j}} \right\} \tag{48}$$

subject to the conditions

$$\sum_{a=0}^{M} x_{2i-1_a}\, a + \sum_{a=0}^{M} x_{2i,a} \geqq L_i / S^h \sum_{a=0}^{M} x_{oa} \tag{49}$$

$(i = 1, \cdots, N)$,

$$S^h \sum_{a=0}^{M} x_{oa} = 1. \tag{50}$$

$$\sum_{R=0}^{M} \sum_{a=0}^{M} q_{ij}\,(a)\, x_{ka} = \sum_{a=0}^{M} x_{ia} \qquad (i = 0, 1, \cdots, M), \tag{51}$$

$$\sum_{i=0}^{M} \sum_{a=0}^{M} x_{ia} = 1, \tag{52}$$

$$\sum_{k=1}^{M} \sum_{a=0}^{M} q_{k_j}\,(a)\, x_{ika} = \sum_{a=0}^{M} x_{ija}\,, \tag{53}$$

$(i, j = 1, \cdots. M)$, and x_{ij}, $x_{ija} \geqq 0$, all i, j, and a.

The constraints (49), (50), (51), and (52) follow in the same manner as the constraints in Model I. (53) follows by using the transformations (46) and (45), using the fact that for any patrol policy P^M,

$$\bar{\pi}_{ij}(P^M) = \sum_{a=0}^{M} x_{ija} \qquad (i, j = 1, \cdots, M),$$ and the transition

probabilities in $Q^{P^M} = (P_{ij}^{P^M})$ satisfy the conditions

$$P_{ij}^{P^M} = \sum_{a=0}^{M} q_{ij}(a)\, D_{ia}^{P^M} \qquad (i, j = 1, \cdots, M).$$

Model II** is a <u>non-linear</u> fractional programming problem with $4N^2+3N+1$ constraints (N = the number of streets in the sector, M = 2N). Fortunately, the denominator of the objective function (the term being minimized) is convex and quadratic, and the numerator is linear. Hence the problem can be solved by using existing quadratic programming computer codes by manipulating the objective function following the procedures suggested by W. Dinklebach, see references [21] and [22].

Next, assume we have solved II** obtaining an optimal solution, say $\{x_{ia}^o: i, a = 0, 1, \cdots, M\}$ and $\{x_{ija}^o: i, j = 1, \cdots, M; a = 0, \cdots, M\}$. Then the optimal patrol policy $\{D_{ia}: i, a = 0, 1, \cdots, M\}$ can be computed using the relation

$$D_{ia}^o = \frac{x_{ia}^o}{\sum_{j=0}^{M} x_{ij}^o} \qquad i, a = 0, 1, \cdots, M$$

and, finally, the optimal patrol matrix $P^o = (P_{ij}^o)$,

$$P_{ij}^o = \sum_{a=0}^{M} q_{ij}(a)\, D_{ia}^o \quad .$$

In Model II we have distinguished between directions of patrol on a street. For example, some streets in the sector may be one-way, U-turns may not be allowed, the type of preventive patrol (one-man, two-man, etc.) may concentrate on one side of a particular street, etc. However, in Model II, considering the nature of the exponential return functions, it may be appropriate in certain cases to define new random variables, $\overline{N}_1(T_o)$, $\overline{N}_2(T_o)$, $\cdots$, $\overline{N}_N(T_o)$ by setting

$$\overline{N}_i(T_o) = N_{2i-1}(T_o) + N_{2i}(T_o)$$

$i = 1, \cdots, N$, N = number of streets in the sector.

We would then compute the expectation of the random variables $e^{-\varphi_i\left(\overline{N}_i(T_o)\right)}$ for each i (=1, $\cdots$, N) and use the resulting sum as our objective function rather than computing the expectation of $e^{-\varphi_{2i-1}\left(N_{2i-1}(T_o)\right)} + \ell^{-\varphi_{2i}\left(N_{2i}(T_2)\right)}$ as was done in Model II.

Computing the expectations of the random variables $e^{-\varphi_i\left(\overline{N}_i(T_o)\right)}$ can readily be done for any policy P^M in M by defining a new Markov matrix $\overline{P}^{P^M}$ with N + 1 states where each state i in $\overline{P}^{P^M}\left(i = 1, \cdots, N\right)$ is obtained by "lumping" states 2i-1 and 2i from the Markov matrix $\left((M + 1) \times (M + 1)\right)$ P^{P^M}. See [23].

We assumed in the development of models I and II that the patrol car travels at a constant speed through each street in the sector and also, calls for service are not allowed. It may be appropriate in some cases to assume that the speed of patrol is a random variable for each street in the sector. This case can be handled by the use of semi-Markov programming [24].

Calls for service within the sector as well as in other sectors can be handled by appropriately enlarging the state space and assuming random handling and travel times for the added locations. This case can also be modelled using semi-Markov programming.

For both of the preceding cases, the new models would still exhibit the same basic structure as in Models I and II.

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