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MULTIVARIATE ECONOMIC CONTROL CHARTS FOR THE MEAN

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ABSTRACT

In this paper, we have determined the economic parameters (sample size and the control chart constant) for two quality characteristics by extending Page's scheme of minimizing L_1 (the average run length of an out of control process) for a large fixed value of L_0 (the average run length of an in control process). One result is that, for correlated quality characteristics, the sample size needed to detect shifts in the levels of these characteristics does not always decrease as the magnitudes of the shifts that are important to detect increase. Another is that in general a smaller sample size is needed for two quality characteristics than for one.

INTRODUCTION

Since the inception of the Shewhart control chart in 1931, numerous publications have dealt with the exposition and application of these charts. More recently, increasing attention has focused on the economic design of these charts. One of the pioneering works in this area was that of Page in 1953. Page discussed the choice of sample size and control limits for controlling the mean of a univariate normal population using the scheme of minimizing the average run length of an out of control process, where the average run length is defined as the average number of articles inspected between two successive occasions when rectifying action is taken. Although this is a very simple type of economic control chart, it is perhaps the most valuable because of its ease of understanding and implementation.

In today's complex society, the quality of each manufactured item depends upon several (p) quality characteristics. Early attempts to monitor the quality of the p characteristics were ad hoc in that p univariate control charts were constructed. If a point fell out of the control limits on any of the charts, the process was judged to be out of control. The shortcomings of this approach have been frequently pointed out in the literature, and publications have appeared which indicate the correct multivariate procedures to be adopted.

Although quality control management has long recognized the need to implement multivariate control procedures, quite frequently thy have failed to do so. One of the reasons is that two crucial questions remain unanswered: (i) how large a sample should be selected? (ii) what value should be used for the control chart constant? This paper answers both of those previously unaddressed questions by extending Page's scheme to the multivariate case. Furthermore, it also allows the quality control decision maker to determine the reduction over other methods in the amount of scrap produced before an out of control state is detected.

ONE QUALITY CHARACTERISTIC

When there is only one quality characteristic (X) which is normally distributed with standard values specified for the process mean (μ_0) and standard deviation (σ_I) and successive random samples of size n are generated from this process, the control chart limits are of the form $\mu_0 \pm B(\sigma_I/\sqrt{n})$, where customarily $B = z_{\alpha/2} = 3.0$. In accordance with Page [2], let m denote the true value of the process mean which may vary from period to period. However, σ_I remains constant. Thus, $X \sim N(m, \sigma_I^2)$.

Let P(m) denote the probability that a given sample yields an $\bar{x}$ outside the control limits when m is the process mean. Then

$$P(m) = P(\bar{X} > \mu_0 + B\ \sigma_I/\sqrt{n}\ |m) + P(\bar{X} < \mu_0 - B\ \sigma_I/\sqrt{n}\ |m)$$

$$= P(Z > B + (\mu_0 - m)/(\sigma_I/\sqrt{n})) + P(Z < -B + (\mu_0 - m)/(\sigma_I/\sqrt{n})) \quad (1)$$

Let Y be a random variable denoting the number of samples up to and including the first one for which an $\bar{x}$ indicates an out of control process. Then Y is a geometric random variable with parameter P(m). Specifically,

$$p_Y(y) = P(m)[1 - P(m)]^{y-1}, \quad y = 1,2,\ldots$$
$$= 0 \quad , \text{otherwise}$$

It is well known that $E(Y) = 1/P(m)$.

Page defines the <u>average run length</u> (L) as the average number of articles inspected between two successive occasions when rectifying action is taken. For constant m,

$$L = nE(Y) = n/P(m),$$

which is the sample size per sample times the average number of samples up to and including the first one out of control.

Let L_0 denote the average run length when $m = \mu_0$. Since $P(\mu_0) = 2\ \Phi(-B)$, it follows that

$$L_0 = n/[2\ \Phi(-B)]. \quad (2)$$

Let $k > 0$ be a value such that "a shift in the mean m of amount equal to or greater than $k\sigma_I$ is serious and we desire tht such a shift should be detected as soon as possible after it has occurred." Define L_1 to be the average run length when $m = \mu_0 + k\sigma_I$. Since

$$P(\mu_0 + k\sigma_I) = \Phi(-B + k\sqrt{n}) + \Phi(-B - k\sqrt{n}),$$

it follows that

$$L_1 = n/[\Phi(-B + k\sqrt{n}) + \Phi(-B - k\sqrt{n})]. \tag{3}$$

Page's scheme for determining B and n is to choose that inspection scheme such that L_1 is minimized for some given large value of L_0 and fixed k. By rewriting equation (2) as $n = 2\ L_0\ \Phi(-B)$ and substituting this result into equation (3), we see that

$$L_1 = \frac{2\ L_0\ \Phi(-B)}{\Phi(-B + k\sqrt{2\ L_0\ \Phi(-B)}) + \Phi(-B - k\sqrt{2\ L_0 \cdot \Phi(-B)})} \tag{4}$$

The problem is to find B which minimizes L_1 for fixed L_0 and k. This B is then used to find n from the equation $n = 2\ L_0\ \Phi(-B)$. By using a computer search routine, Page constructed tables of n, B, and L_1 for L_0 = 2,000, 5,000, 10,000, 15,000, 20,000, 40,000, and 60,000 and k = (0.2)(0.1)(1.8), where (0.1) denotes the step size of k.

A computer program was written to duplicate Page's results, and, in order to facilitate later comparisons, the output is given in Table 1 for L_0 = 10,000 and k = 0.2 (0.2) 1.8.

These results correspond almost exactly with those of Page. By inspecting the table, we see that, for a fixed L_0, n decreases as k increases. This is intuitively appealing since it says that a larger sample is needed to detect a small shift while a smaller sample will suffice for a large shift. Perhaps the most surprising result of Table 1 is that the control chart constant is quite frequently less than 3.0, the traditional value. For example, when $L_0 = 10,000$, it is only when $k \geq 0.80$ that $B \geq 3.0$. Thus, Page's scheme calls for tighter than usual control limits and larger than usual sample sizes to detect small shifts.

Table 1. Values of n, $\chi^2_{1,\alpha} = B^2$, and L_1 for fixed L_0 and k

One Characteristic, Independent Observations					
L_0	k	n	$\chi^2_{1,\alpha}$	B	L_1
10000	.20	187	5.528	2.351	287.8
10000	.40	65	7.405	2.721	93.8
10000	.60	34	8.579	2.929	47.5
10000	.80	21	9.459	3.076	29.1
10000	1.00	14	10.199	3.194	19.8
10000	1.20	11	10.635	3.261	14.4
10000	1.40	8	11.241	3.353	11.0
10000	1.60	6	11.774	3.431	8.7
10000	1.80	5	12.110	3.480	7.1

MULTIPLE QUALITY CHARACTERISTICS

This section extends the results of the previous section by allowing the quality of each item to be governed by more than one quality characteristic. In order to do this, we need to recall certain underlying principles of multivariate statistical quality control. We will only consider two quality characteristics although the results are easily extended.

Suppose $\underset{\sim}{X}_1 = [X_{11}, \underset{\sim}{X}_{21}]^t, \ldots, \underset{\sim}{X}_n = [X_{1n}, X_{2n}]^t$ is a random sample of size n from a bivariate normal process with mean vector $\underset{\sim}{m}$ and known variance-covariance matrix Σ (possibly obtained from a large amount of past data). Let $\underset{\sim}{\mu}_0 = [\mu_1^0, \mu_2^0]^t$ denote the nominal value of the process mean. To maintain statistical control over $\underset{\sim}{\mu}_0$, the vector of sample means ($\underset{\sim}{\bar{x}} = [\bar{x}_1, \bar{x}_2]^t$) is calculated and it is necessary to determine whether $n(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu}_0)^t \Sigma^{-1}(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu}_0)$ exceeds the upper control limit ($\chi^2_{2,\alpha}$). Note that this is equivalent to testing $H_0: \underset{\sim}{\mu} = \underset{\sim}{\mu}_0$ vs. $H_1: \underset{\sim}{\mu} \neq \underset{\sim}{\mu}_0$ with Σ known. Additional background on the multivariate quality control problem can be found in Alt [1].

Page's procedure can be extended to this multivariate case by determing $P(\underset{\sim}{m})$ where

$$P(\underset{\sim}{m}) = P[n(\underset{\sim}{\bar{X}} - \underset{\sim}{\mu}_0)^t \Sigma^{-1}(\underset{\sim}{\bar{X}} - \underset{\sim}{\mu}_0) > \chi^2_{2,\alpha} \,|\underset{\sim}{m}], \tag{5}$$

which is the probability that the statistic plots out of control when the true process mean is $\underset{\sim}{m}$. If $\underset{\sim}{X} \sim N_2(\underset{\sim}{m}, \Sigma)$ and hence $\underset{\sim}{\bar{X}} \sim N_2(\underset{\sim}{m}, \Sigma/n)$, then if follows that

$$n(\underset{\sim}{\bar{X}} - \underset{\sim}{\mu}_0)^t \Sigma^{-1}(\underset{\sim}{\bar{X}} - \underset{\sim}{\mu}_0) \sim \chi'^2_{2,\lambda}$$

where the prime denotes the noncentral chi-square random variable and the noncentrality parameter

$\lambda = n(\underset{\sim}{m} - \underset{\sim}{\mu}_0)^t \, \Sigma^{-1}(\underset{\sim}{m} - \underset{\sim}{\mu}_0)$. Thus

$$P(\underset{\sim}{m}) = P(\chi'^2_{2,\lambda} > \chi^2_{2,\alpha} \mid \underset{\sim}{m}).$$

When $\underset{\sim}{m} = \underset{\sim}{\mu}_0$, $\lambda = 0$,

$$P(\underset{\sim}{\mu}_0) = P(\chi_2^2 > \chi^2_{2,\alpha}),$$

and

$$L_0 = n/P(\chi_2^2 > \chi^2_{2,\alpha}). \tag{6}$$

In order to measure departures of $\underset{\sim}{m}$ from $\underset{\sim}{\mu}_0$, it is necessary to account for the possible departure of each component of $\underset{\sim}{m}$. This is accomplished by introducing the (2 x 1) vector $\underset{\sim}{\sigma}$ where

$$\underset{\sim}{\sigma}^t = [k_1\sigma_1, \; k_2\sigma_2],$$

and letting $\underset{\sim}{m} = \underset{\sim}{\mu}_0 + \underset{\sim}{\sigma}$. We require that at least one $k_i > 0$. Thus, although we wish to simultaneously control the mean vector of several variables, it may be necessary to detect a shift in only one of these variables. When $\underset{\sim}{m} = \underset{\sim}{\mu}_0 + \underset{\sim}{\sigma}$,

$$\lambda = n \, \underset{\sim}{\sigma}^t \, \Sigma^{-1} \, \underset{\sim}{\sigma} = n(1 - \rho^2)^{-1}(k_1^2 - 2\rho k_1 k_2 + k_2^2)$$

$$P(\underset{\sim}{\mu}_0 + \underset{\sim}{\sigma}) = P(\chi'^2_{2,\lambda} > \chi^2_{2,\alpha}),$$

and

$$L_1 = n/P(\chi'^2_{2,\lambda} > \chi^2_{2,\alpha}). \tag{7}$$

As with the univariate case, that inspection scheme will be chosen which minimizes L_1 for some given large value of L_0 and fixed $\underset{\sim}{k} = [k_1, k_2]^t$. However, in the multivariate case, we must also fix ρ which is the correlation between quality characteristics X_1 and X_2. By rewriting equation (6) as $n = L_0 \, P(\chi_2^{\,2} > \chi_{2,\alpha}^{2})$ and substituting this result into equation (7), we see that

$$L_1 = \frac{L_0 \, P(\chi_2^{\,2} > \chi_{2,\alpha}^{2})}{P(\chi'^{\,2}_{2,\lambda} > \chi_{2,\alpha}^{2})} \qquad (8)$$

For fixed L_0, k, and ρ we seek that $\chi_{2,\alpha}^{2}$ and n which minimizes L_1 as stated in equation (8). One difficulty in doing this is the need for evaluating the denominator of equation (8), which is the complementary cumulative noncentral chi-square distribution function evaluated at $\chi_{2,\alpha}^{2}$. To accomplish this, we use the remarkably accuarate approximation of Sankaran [3]. The search routine used to find the minimum L_1 is a modified version of the success-failure method as described by Dixon. The minimization of L_1 was investigated for $L_0 = 10{,}000$, $\rho = (-0.8)(0.4)(+0.8)$, $k_1 = 0.2, 0.6, 1.0$ and $k_2 = 0.0, 0.2, 0.6, 1.0$. The results are printed in Table 2.

Upon first glancing at Table 2, it appears that, for a fixed L_0 and ρ, n decreases as k_1 and k_2 increase. Again, this is intuitively appealing, for the magnitude of the required sample size should indeed decrease as the

Table 2

Economic Parameters for Two Quality Characteristics (L_0 = 10,000)

		ρ = -0.8			ρ = -0.4			ρ = 0.0			ρ = 0.4			ρ = 0.8		
k_1	k_2	n	$\chi^2_{2,\alpha}$	L_1*	n	$\chi^2_{2,\alpha}$	L_1*	n	$\chi^2_{2,\alpha}$	L_1*	n	$\chi^2_{2,\alpha}$	L_1*	n	$\chi^2_{2,\alpha}$	L_1*
0.2	0.0	103	9.15	148 [1]	199	7.83	295 [9]	227	7.56	339	199	7.83	295 [9]	103	9.15	148 [1]
	0.2	36	11.25	50 [2]	89	9.44	127	133	8.64	194	173	8.11	254	209	7.74	311
	0.6	11	13.62	15 [3]	26	11.90	36 [10]	36	11.25	50 [2]	39	11.09	55 [16]	27	11.82	36 [19]
	1.0	5	15.20	7 [4]	12	13.45	17 [11]	16	12.87	22 [15]	16	12.87	22 [15]	9	14.02	12 [18]
0.6	0.0	17	12.75	23 [5]	34	11.37	47 [12]	40	11.09	55 [16]	34	11.37	47 [12]	17	12.75	23 [5]
	0.2	11	13.62	15 [3]	26	11.90	36 [10]	36	11.25	50 [2]	39	11.09	55 [16]	27	11.82	36 [19]
	0.6	6	14.81	7 [6]	14	13.14	19 [13]	22	12.24	30	29	11.68	40	36	11.25	50 [2]
	1.0	3	16.22	4 [7]	9	14.02	11 [14]	13	13.29	17 [17]	16	12.87	22 [15]	15	13.00	21 [20]
1.0	0.0	7	14.51	9 [8]	14	13.14	19 [13]	17	12.75	23 [5]	14	13.14	19 [13]	7	14.51	9 [8]
	0.2	5	15.20	7 [4]	12	13.45	17 [11]	16	12.87	22 [15]	16	12.87	22 [15]	9	14.02	12 [18]
	0.6	3	16.22	4 [7]	9	14.02	11 [14]	13	13.29	17 [17]	16	12.87	22 [15]	15	13.00	21 [20]
	1.0	2	17.00	3	6	14.81	8 [6]	9	14.02	12 [18]	12	13.45	17 [11]	15	13.00	21 [20]

*The numerical superscripts indicate those entries which have the same values of the economic parameters n, $\chi^2_{2,\alpha}$, and L_1.

magnitudes of the shifts which are important to detect increase. Usually, n is much larger for small k_1 and k_2 = 0.0 than for other values of k_2. The interpretation of $k_2 = 0.0$ is that it is important to detect a shift of zero magnitude in the second component, or an "infinitesimally small" shift. This accounts for the rather large sample sizes in this case. However, further inspection of Table 2 shows that it is not always true that n decreases as k_1 and k_2 increase for fixed L_0 and ρ. While this is true for $\rho \leq 0$ and also for $\rho = 0.4$ when k_1 is small, it is not true for the other values of k_1 and $\rho = 0.4$, nor is it ever true when $\rho = 0.8$. Thus, for a relatively large positive correlation, the sample size needed to detect large positive shifts is larger than the sample sizes needed for smaller positive shifts. An explanation of this is provided by examining the noncentrality parameter λ, which is a generalized measure of distance of how far the true mean is from the nominal value. Fix $\rho = +0.4$ and $k_1 = 0.6$. When $k_2 = 0.2$, $\lambda = (n/.84)(.304)$; when $k_2 = 0.0$, $\lambda = (n/.84)(.360)$; and, when $k_2 = 0.6$, $\lambda = (n/.84)(.432)$. Inspection of Table 2 shows that, for the (k_1, k_2) pairs investigated, the largest sample size (35) occurred with the smallest value of the noncentrality parameter (.304), the next largest sample size (30) occurred with the next to the smallest value of the noncentrality parameter (.36), and the smallest sample size (26) occurred with the largest value of the noncentrality parameter. Thus, when the generalized measure of distance (λ) between the true mean and the nominal value is small, it is to be expected that a larger sample size will be needed to detect such a small shift.

Let us now compare n for positive ρ with n for negative ρ. It is to be expected that both n's will be equal when $k_2 = 0$ since $\lambda = n(1 - \rho^2)^{-1} k_1^2$ and the sign of ρ is lost through the squaring operation. However, for fixed k_1 and k_2, n is always much smaller for $\rho < 0$. However, this is not to imply that one should try to choose negatively correlated characteristics as opposed to positively correlated characteristics. The stated phenomenon occurs because we are looking at positive shifts ($k_1 > 0$, $k_2 > 0$) instead of negative shifts ($k_1 < 0$, $k_2 < 0$). Thus for $\rho < 0$ and $k_1 > 0$, $k_2 > 0$, the generalized distance measure (λ) is larger than for $\rho > 0$ and $k_1 > 0$, $k_2 > 0$.

One additional topic of interest is how does the required sample size for two quality characteristics compare with the sample size for one quality characteristic (Table 1)? Some idea of this behavior is obtained by letting $\rho = 0.0$. Thus, $\lambda = n(k_1^2 + k_2^2)$. Now, when $k_2 = 0$, λ reduces to the univariate noncentrality parameter $n\,k^2$. However, the control limit will still be $\chi^2_{2,\alpha}$. Tables 1 and 2 show that, for $\rho = 0.0$, $k_2 = 0.0$ and fixed L_0 and k_1, the required sample size is larger for two quality characteristics than for one quality characteristic with this difference becoming smaller as k_1 increases. Furthermore, as soon as k_2 becomes positive, n for p = 2 is usually much smaller than for p = 1. Thus, an economical sample size is not an unusual result when two quality characteristics are used as opposed to one. As a final point of interest, note that the maximum n in Table 2 occurs for $\rho = 0.0$, $k_1 = 0.2$, and $k_2 = 0.0$. This is the one case where the required sample size for p = 1 (Table 1) is considerably smaller than for p = 2 (Table 2).

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